THE CALCULUS

A SERIES OF MATHEMATICAL TEXTS

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EARLE RAYMOND HEDRICK

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THE CALCULUS

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PREFACE

THE significance of the Calculus, the possibility of applying it in other fields, its usefulness, ought to be kept constantly and vividly before the student during his study of the subject, rather than be deferred to an uncertain future.

Not only for students who intend to become engineers, but also for those planning a profound study of other sciences, the usefulness of the Calculus is universally recognized by teachers; it should be consciously realized by the student himself. It is obvious that students interested primarily in mathematics, particularly if they expect to instruct others, should recognize the same fact.

To all these, and even to the student who expects only general culture, the use of certain types of applications tends to make the subject more real and tangible, and offers a basis for an interest that is not artificial. Such an interest is necessary to secure proper attention and to insure any real grasp of the essential ideas.

For this reason, the attempt is made in this book to present as many and as varied applications of the Calculus as it is possible to do without venturing into technical fields whose subject matter is itself unknown and incomprehensible to the student, and without abandoning an orderly presentation of fundamental principles.

The same general tendency has led to the treatment of topics with a view toward bringing out their essential usefulness. Thus the treatment of the logarithmic derivative is vitalized by its presentation as the relative rate of change of a quantity; and it is fundamentally connected with the important "compound interest law," which arises in any phenomenon in

which the relative rate of increase (logarithmic derivative) is constant.

Another instance of the same tendency is the attempt, in the introduction of the precise concept of curvature, to explain the reason for the adoption of this, as opposed to other simpler but cruder measures of bending. These are only instances, of two typical kinds, of the way in which the effort to bring out the usefulness of the subject has influenced the presentation of even the traditional topics.

Rigorous forms of demonstration are not insisted upon, especially where the precisely rigorous proofs would be beyond the present grasp of the student. Rather the stress is laid upon the student's certain comprehension of that which is done, and his conviction that the results obtained are both reasonable and useful. At the same time, an effort has been made to avoid those grosser errors and actual misstatements of fact which have often offended the teacher in texts otherwise attractive and teachable.

Thus a proof for the formula for differentiating a logarithm is given which lays stress on the very meaning of logarithms; while it is not absolutely rigorous, it is at least just as rigorous as the more traditional proof which makes use of the limit of $(1+1/n)^n$ as n becomes infinite, and it is far more convincing and instructive. The proof used for the derivative of the sine of an angle is quite as sound as the more traditional proof (which is also indicated), and makes use of fundamentally useful concrete concepts connected with circular motion. These two proofs again illustrate the tendency to make the subject vivid, tangible, and convincing to the student; this tendency will be found to dominate, in so far as it was found possible, every phase of every topic.

Many traditional theorems are omitted or reduced in importance. In many cases, such theorems are reproduced in exercises, with a sufficient hint to enable the student to master them. Thus Taylor's Theorem in several variables, for which

wide applications are not apparent until further study of mathematics and science, is presented in this manner.

On the other hand, many theorems of importance, both from mathematical and scientific grounds, which have been omitted traditionally, are included. Examples of this sort are the brief treatment of simple harmonic motion, the wide application of Cavalieri's theorem and the prismoid formula, other approximation formulas, the theory of least squares (under the head of exercises in maxima and minima), and many other topics.

The Exercises throughout are colored by the views expressed above, to bring out the usefulness of the subject and to give tangible concrete meaning to the concepts involved. Yet formal exercises are not at all avoided, nor is this necessary if the student's interest has been secured through conviction of the usefulness of the topics considered. Far more exercises are stated than should be attempted by any one student. This will lend variety, and will make possible the assignment of different problems to different students and to classes in successive years. It is urged that care be taken in selecting from the exercises, since the lists are graded so that certain groups of exercises prepare the student for other groups which follow; but it is unnecessary that all of any group be assigned, and it is urged that in general less than half be used for any one student. Exercises that involve practical applications and others that involve bits of theory to be worked out by the student are of frequent occurrence. These should not be avoided, for they are in tune with the spirit of the whole book; great care has been taken to select these exercises to avoid technical concepts strange to the student or proofs that are too difficult.

An effort is made to remove many technical difficulties by the intelligent use of tables. Tables of Integrals and many other useful tables are appended; it is hoped that these will be found usable and helpful.

Parts of the book may be omitted without destroying the essential unity of the whole. Thus the rather complete treat-

ment of Differential Equations (of the more elementary types) can be omitted. Even the chapter on Functions of Several Variables can be omitted, at least except for a few paragraphs, without vital harm; and the same may be said of the chapter on Approximations. The omission of entire chapters, of course, would only be contemplated where the pressure of time is unusual; but many paragraphs may be omitted at the discretion of the teacher.

Although care has been exercised to secure a consistent order of topics, some teachers may desire to alter it; for example, an earlier introduction of transcendental functions and of portions of the chapter on Approximations may be desired, and is entirely feasible. But it is urged that the comparatively early introduction of Integration as a summation process be retained, since this further impresses the usefulness of the subject, and accustoms the student to the ideas of derivative and integral before his attention is diverted by a variety of formal rules.

Purely destructive criticism and abandonment of coherent arrangement are just as dangerous as ultra-conservatism. This book attempts to preserve the essential features of the Calculus, to give the student a thorough training in mathematical reasoning, to create in him a sure mathematical imagination, and to meet fairly the reasonable demand for enlivening and enriching the subject through applications at the expense of purely formal work that contains no essential principle.

E. W. DAVIS,
W. C. BRENKE,
E. R. HEDRICK, EDITOR.

JUNE, 1912.

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"Be not the first by whom the new is tried,
Nor yet the last to lay the old aside."

— POPE.

THE CALCULUS

CHAPTER I

FUNCTIONS

1. Dependence. There are countless instances in which one quantity depends upon another. The speed of a body falling from rest depends upon the time it has fallen. One's income from a given investment depends upon the amount invested and the rate of interest realized. The crops depend upon rainfall, soil fertility and proper cultivation.

In mathematics we usually deal with quantities that are definitely and completely determined by certain others. Thus the area A of a square is determined precisely when the length s of its side is given: $A = s^2$; the volume of a sphere is $4\pi r^3/3$; the force of attraction between two bodies is $k \cdot m \cdot m'/d^2$, where m and m' are their masses, d the distance between them, and k a certain number given by experiment. **The Calculus** is the study of the relations between such interdependent quantities, with special reference to their rates of change.

2. Variables. Constants. Functions. A quantity which may change is called a **variable**. The quantities mentioned in § 1, except k and π, are examples of variables.

A quantity which has a fixed value is called a **constant**. Examples of constants are ordinary numbers: 1, $\sqrt{2}$, -7, $2/3$, π, $30°$, $\log 5$, and the number k in § 1.

If one variable y depends on another variable x, so that y is determined when x is known, y is said to be a **function** of x.

The variable x, thus thought of as determining the other, is called the **independent variable**; the other variable y is called the **dependent variable.** Thus, in § 1, the area A of a square is a function, $A = s^2$, of the side s.

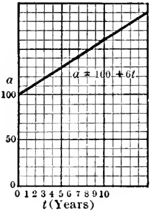


FIG. 1.

In **Algebra** we learn how to express such relations by means of equations.

In **Analytic Geometry** such relations are represented graphically. For example, if the principal at simple interest is a fixed sum p and if the interest rate r also is fixed, then the amount a, of principal and interest, varies solely with (is a function of) the time t that the principal has been at interest. In fact, if $p = 100$ and $r = 6\%$,

$$a = p + ptr = 100 + 6t.$$

This is represented graphically in Fig. 1. In practice fractional parts of a day are neglected.

The relation $A = s^2$ of § 1 is represented in Fig. 2.

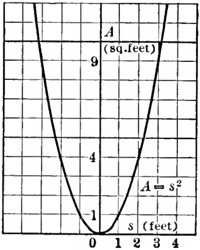


FIG. 2.

EXERCISES I.—FUNCTIONS AND GRAPHS

Represent graphically the following:—

1. $a = 100 + 3t$, $a = 300 + 4t$, $a = 150 + 7t$.

2. The number of feet f in terms of the number of yards y in a given length is given by the equation $f = 3y$.

3. The temperature in degrees Fahrenheit, F, is 32 more than 9/5 the temperature in degrees Centigrade, C.

4. The distance s that a body falls from rest in a time t is given by $s = 16t^2$. (Measure t horizontally and s vertically downward.)

5. (a) $y = x^2 + 3x + 1$. (b) $y = 2x^2 - 5x$.

(c) $y = x^3 + 2$. (d) $y = \dfrac{1}{x+1}$.

(e) $y = \dfrac{x-1}{x+2}$. (f) $y = \dfrac{x^2 + 2x + 3}{x - 5}$.

6. The volume v of a fixed quantity of gas at a constant temperature varies inversely as the pressure p upon the gas.

7. The amount of \$1.00 at compound interest at 10% per annum for t years is $a = (1 + 1/10)^t$.

8. The area A of an equilateral triangle is a function of its side s. Determine this function, and represent the relation graphically. Express the side in terms of the area.

9. Determine the area a of a circle in terms of its radius r. Determine the radius in terms of the area.

10. The radius, surface, and volume of a sphere are functionally related. Find the equations connecting each pair. Also express each of the three as a function of the circumference of a great circle of the sphere.

11. The area A bounded by the straight line $y = ax + b$, the ordinate y, and the axes, is a function of x. Determine it; and also express y as a function of the area.

3. The Function Notation. A very useful abbreviation for functions consists in writing $f(x)$ (read f of x) in place of the given expression.

Thus if $f(x) = x^2 + 3x + 1$, we may write $f(2) = 2^2 + 3 \cdot 2 + 1 = 11$, that is, the value of $x^2 + 3x + 1$ when $x = 2$ is 11. Likewise $f(3) = 19$, $f(-1) = -1$, $f(0) = 1$, and so on. $f(a) = a^2 + 3a + 1$. $f(u + v) = (u + v)^2 + 3(u + v) + 1$.

Other letters than f are often used, to avoid confusion, but f is used most often, because it is the initial of the word *function*. Other letters than x are often used for the variable. In any case, given $f(x)$, to find $f(a)$, simply substitute a for x in the given expression.

EXERCISES II. — SUBSTITUTION FUNCTION NOTATION

1. If $f(x) = x^2 - 5x + 2$ find $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(0)$, $f(-1)$, $f(-2)$. From these values (and others, if needed) draw the graph of

the curve $y = f(x)$. Mark its lowest point, and estimate the values of x and y there.

2. Proceed as in Ex. 1 for each of the following functions, using the function notation in calculating values; mark the highest and lowest points if any exist, and estimate the values of x and y at these points.

(*a*) $x^3 - 2x + 4$. (*b*) $3x^2 - 2x + 1$. (*c*) $\dfrac{x+1}{2x-3}$. (*d*) $\dfrac{1}{x+1} + \dfrac{2}{x-1}$.

(*e*) $y = \sin x$, taking $x = \pi/6,\ \pi/4,\ \pi/2,\ 3\pi/4,\ \pi,\ 0,\ -\pi/2$.

(*f*) $y = \log_{10} x$, taking $x = 1,\ 2,\ 10,\ 1/10,\ 1/100$.

3. If $f(x) = x^4 - 5x^3 + 3x^2 - 2x + 3$, calculate $f(1)$, $f(4)$, $f(5)$. Hence show that one solution of the equation $f(x) = 0$ is $x = 1$; and that another solution lies between 4 and 5.

(This work is simplified by using the theorem that $f(a)$ is equal to the remainder obtained by dividing $f(x)$ by $(x-a)$; and by using synthetic division.)

4. If $f(x) = 2x^2 - 3x + 5$, show that $f(a) = 2a^2 - 3a + 5$, $f(m+n) = 2(m+n)^2 - 3(m+n) + 5$; find $f(a-b)$, $f(a+2b)$, $f(a/b)$.

5. If $f(x) = x^2 + 3$ and $\phi(x) = 3x + 1$, show that $f(1) = \phi(1)$ and $f(2) = \phi(2)$. Show that $f(3) > \phi(3)$. Draw $y = f(x)$ and $y = \phi(x)$.

6. In Ex. 5, draw the curve $y = f(x) - \phi(x)$. Mark the points where $f(x) - \phi(x) = 0$. Mark the lowest point.

7. If $f(x) = -2x^2 + 1$ and $\phi(x) = x^2 + 2x + 4$, find the value for which $f(x) = \phi(x)$ by use of $f(x) - \phi(x)$. Sketch all of the curves $y = f(x)$, $y = \phi(x)$, $y = f(x) - \phi(x)$.

8. If $f(x) = \sin x$ and $\phi(x) = \cos x$, show that $[f(x)]^2 + [\phi(x)]^2 = 1$; $f(x) \div \phi(x) = \tan x$; $f(x+y) = f(x)\,\phi(y) + f(y)\,\phi(x)$; $\phi(x+y) = ?$; $f(x) = \phi(\pi/2 - x)$; $\phi(x) = f(\pi/2 - x) = -\phi(\pi - x)$; $f(-x) = -f(+x)$: $\phi(-x) = \phi(x)$.

9. If $f(x) = \log_{10} x$, show that

$$f(x) + f(y) = f(x \cdot y); \qquad f(x^2) = 2f(x);$$

$$f(m/n) - f(n/m) = 2f(m) - 2f(n); \qquad f(m/n) + f(n/m) = 0.$$

10. If $f(x) = \tan x$, $\phi(x) = \cos x$, draw the curves $y = f(x)$, $y = \phi(x)$, $y = f(x) - \phi(x)$. Mark the points where $f(x) = \phi(x)$ and estimate the values of x and y there.

11. Taking $f(x) = x^2$, compare the graph of $y = f(x)$ with that of $y = f(x) + 1$, and with that of $y = f(x+1)$.

12. Taking any two curves $y=f(x)$, $y=\phi(x)$, how can you most easily draw $y=f(x)-\phi(x)$? $y=f(x)+\phi(x)$? Draw $y=x^2+1/x$.

13. How can you most easily draw $y=f(x)+5$? $y=f(x+5)$? assuming that $y=f(x)$ is drawn.

14. Draw $y=x^2$ and show how to deduce from it the graph of $y=2x^2$; the graph of $y=-x^2$.

Assuming that $y=f(x)$ is drawn, show how to draw the graph of $y=2f(x)$; that of $y=-f(x)$.

15. From the graph of $y=x^2$, show how to draw the graph of $y=(2x)^2$; that of $y=x^2+2$; that of $y=(x+2)^2$; that of $y=(2x-3)^2$.

16. What change is made in a curve if x, in the equation, is replaced by $-x$? if y by $-y$? if both things are done? Compare the graphs of $y=f(x)$, $y=f(-x)$, $-y=f(x)$; $y=2f(x)$; $y=f(x)+2$.

17. What change is made in a curve if x is replaced by $2x$, $3x$, $x/2$? Compare the graphs of $y=f(x)$, $y=f(2x)$, $y=f(3x)$, $y=f(x/2)$; $y=f(x+2)$.

18. What is the effect upon a curve if, in the equation, x and y are interchanged? Compare the graphs of $y=f(x)$, $x=f(y)$.

19. Plot the following curves: (*a*) $y+2=\sin(3x+2)$, (*b*) $y=x+\sin x$, (*c*) $y=2^x-\sin x$, (*d*) $y=2^x\cos x$, (*e*) $3x+4y=4\sin(4x-3y)$, (*f*) $y=(\cos x)/(2x+3)$, (*g*) $\sin y=\cos 2x$, (*h*) $y=\log_2(x^2+1)$.

20. In polar coördinates (r, θ), what change is made in a curve if, in the equation, θ is replaced by 2θ, if r is replaced by $2r$?

21. What change in θ is equivalent to a change in the sense of r.

22. From the graph of $r=f(\theta)$ derive those of (*a*) $r=f(2\theta)$, (*b*) $r=2f(\theta)$, (*c*) $r=f(-\theta)$, (*d*) $r=-f(\theta)$, (*e*) $r+1=f(\theta)$, (*f*) $r=f(\theta+1)$, (*g*) $r+1=f(\theta+2)$.

Take, for example, $f(\theta)=1$, $f(\theta)=\theta$, $f(\theta)=\sin\theta$, $f(\theta)=2\theta$, $f(\theta)=\arctan\theta$, and draw the variations from the original graph.

23. Plot the following: (*a*) $r=2+3\cos\theta$, (*b*) $r=3+2\cos\theta$, (*c*) $r=2+2\cos\theta$, (*d*) $r=2^\theta$, (*e*) $r^2=a\theta$, (*f*) $\theta=2^r$, (*g*) $\theta^2=ar$, (*h*) $\theta=\sin r$, (*i*) $\theta=\cos r$, (*j*) $\theta=\tan r$, (*k*) $r=\sec(\theta-a)$, (*l*) $\theta=\sec r$.

24. Show how to obtain the graph of $y=A\sin(at+b)$ by suitable modification of the simple sine-curve $y=\sin t$.

25. Draw the graphs from the following equations: (*a*) $2s=e^t+e^{-t}$, (*b*) $2s=e^t-e^{-t}$, (*c*) $s=(e^t+e^{-t})/(e^t-e^{-t})$, (*d*) $s=\sin t+\sin 2t$, (*e*) $s=\sin t+e^{-t}\sin 2t$. Take $e=2.7$, and use logarithms in computations.

CHAPTER II

RATES LIMITS DERIVATIVES

4. Rate of Increase. Slope. In the study of any quantity, its rate of increase (or decrease), when some related quantity changes, is very important for any complete understanding. Thus, the rate of increase of the speed of a boat when the power applied is increased is a fundamental consideration.

Graphically, the rate of increase of y with respect to x is shown by the rate of increase of the height of a curve. If the curve is very flat, there is a small rate of increase; if steep, a large rate.

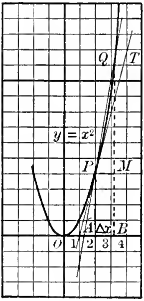


FIG. 3.

The **steepness**, or **slope**, of a curve shows the rate at which the dependent variable is increasing with respect to the independent variable.

When we speak of the slope of a curve at any point P we mean the slope of its tangent at that point. To find this, we must start, as in Analytic Geometry, with a *secant* through P.

Let the equation of the curve, Fig. 3, be $y = x^2$, and let the point P at which the slope is to be found, be the point (2, 4).

Let Q be any other point on the curve, and let Δx represent the difference of the values of x at the two points P and Q.*

* Δx may be regarded as an abbreviation of the phrase, "difference of the x's." The quotient of two such differences is called a **difference quotient.** Notice particularly that Δx does *not* mean $\Delta \times x$. Instead of "difference of the x's," the phrases "change in x" and "increment of x" are often used.

Then, in the figure, $OA = 2$, $AB = \Delta x$, and $OB = 2 + \Delta x$. Moreover, since $y = x^2$ at every point, the value of y at Q is $BQ = (2 + \Delta x)^2$.

The slope S of the secant PQ is the quotient of the differences Δy and Δx:

$$S = \tan \angle MPQ = \frac{\Delta y}{\Delta x} = \frac{MQ}{PM} = \frac{(2 + \Delta x)^2 - 4}{\Delta x} = 4 + \Delta x.$$

The slope m of the tangent at P, that is $\tan \angle MPT$, is the limit of the slope of the secant as Q approaches P.

The slope of the secant is the average slope of the curve between the points P and Q. The slope of the curve at the single point P is the limit of this average slope as Q approaches P.

But, since $S = 4 + \Delta x$, it is clear that the limit of S as Q approaches P is 4, since Δx approaches zero when Q approaches P; hence the slope m of the curve is 4 at the point P.

At any other point the argument would be similar. If the coördinates of P are (a, a^2), those of Q would be $[(a + \Delta x), (a + \Delta x)^2]$; and the slope of the secant would be the difference quotient $\Delta y \div \Delta x$:

$$S = \frac{\Delta y}{\Delta x} = \frac{(a + \Delta x)^2 - a^2}{\Delta x} = \frac{2\,a\Delta x + \overline{\Delta x}^2}{\Delta x} = 2\,a + \Delta x.$$

Hence the slope of the curve at the point (a, a^2) is*

$$m = \lim_{\Delta x \doteq 0} S = \lim_{\Delta x \doteq 0} \Delta y / \Delta x = \lim_{\Delta x \doteq 0} (2\,a + \Delta x) = 2\,a.$$

On the curve $y = x^2$, the slope at any point is numerically twice the value of x.

When the slope can be found, as above, the *equation of the tangent* at P can be written down at once, by Analytic Geometry, since the slope m and a point (a, b) on a line determine its equation:

$$(y - b) = m(x - a).$$

Hence, in the preceding example, at the point $(2, 4)$, where we found $m = 4$, the equation of the tangent is

* Read "$\Delta x \doteq 0$" "as Δx approaches zero." A detailed discussion of limits is given in § 10, p. 16.

$$(y-4)=4(x-2), \quad \text{or} \quad 4x-y=4.$$

At the point (a, a^2) on the curve $y=x^2$, we found $m=2a$; hence the equation of the tangent there is

$$(y-a^2)=2a(x-a), \quad \text{or} \quad 2ax-y=a^2.$$

5. General Rules. A part of the preceding work holds true for any curve, and all of the work is at least similar. Thus, for any curve, the slope is

$$m=\lim_{\Delta x \doteq 0} S=\lim_{\Delta x \doteq 0}(\Delta y/\Delta x);$$

that is, the slope m of the curve is the limit of the difference quotient $\Delta y/\Delta x$.

The changes in various examples arise in the calculation of the difference quotient, $\Delta y \div \Delta x$, or S.

This difference quotient is always obtained, as above, by finding the value of y at Q from the value of x at Q, from the equation of the curve, then finding Δy by subtracting from this the value of y at P, and finally forming the difference quotient by dividing Δy by Δx.

6. Slope Negative or Zero. If the slope of the curve is *negative*, the rate of increase in its height is negative, that is, the height is really *decreasing* with respect to the independent variable.*

If the slope is *zero*, the tangent to the curve is *horizontal*. This is what happens ordinarily at a highest point (maximum) or at a lowest point (minimum) on a curve.†

Example 1. Thus the curve $y=x^2$, as we have just seen, has, at any point $x=a$, a slope $m=2a$. Since m is positive when a is positive, the

* Increase or decrease in the height is always measured as we go toward the right, *i.e.* as the independent variable increases.

† A maximum need not be the highest point on the entire curve, but merely the highest point in a small arc of the curve about that point. See § 37, p. 63. Horizontal tangents sometimes occur without any maximum or any minimum. See § 38, p. 63.

curve is rising on the right of the origin; since m is negative when a is negative, the curve is falling (that is, its height y decreases as x increases) on the left of the origin. At the origin $m = 0$; the origin is the lowest point (a minimum) on the curve, because the curve falls as we come toward the origin and rises afterwards.

Example 2. Find the slope of the curve

(1) $$y = x^2 + 3x - 5$$

at the point where $x = -2$; also in general at a point $x = a$. Use these values to find the equation of the tangent at $x = 2$; the tangent at any point; the maximum or minimum points if any exist.

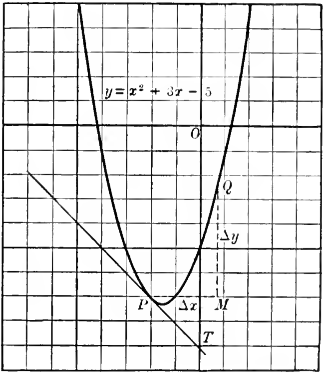


FIG. 4.

When $x = -2$, we find $y = -7$, (P in Fig. 4); taking any second point Q, $(-2 + \Delta x,\ -7 + \Delta y)$, its coördinates must satisfy the given equation, therefore

(2) $$-7 + \Delta y = (-2 + \Delta x)^2 + 3(-2 + \Delta x) - 5,$$

or

(3) $$\Delta y = -4\,\Delta x + \overline{\Delta x}^2 + 3\,\Delta x = -\Delta x + \overline{\Delta x}^2,$$

where $\overline{\Delta x}^2$ means the square of Δx. Hence the slope of the secant PQ is

(4) $$S = \Delta y/\Delta x = -1 + \Delta x.$$

The slope m of the curve is the limit of S as Δx approaches zero; *i.e.*

(5) $$m = \lim_{\Delta x \doteq 0} S = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} (-1 + \Delta x) = -1.$$

It follows that the equation of the tangent at $(-2, -7)$ is

(6) $$(y + 7) = -1(x + 2), \text{ or } x + y + 9 = 0.$$

Likewise, if we take the point P (a, b) in any position on the curve whatsoever, the equation (1) gives

(7) $$b = a^2 + 3a - 5.$$

Any second point Q has coordinates $(a + \Delta x, b + \Delta y)$ where Δx and

Δy are the differences in x and in y, respectively, between P and Q. Since Q also lies on the curve, these coördinates satisfy (1):

(8) $$b + \Delta y = (a + \Delta x)^2 + 3(a + \Delta x) - 5.$$

Subtracting the equation (7) from (8),

$$\Delta y = 2a\Delta x + \overline{\Delta x}^2 + 3\Delta x, \quad \text{whence} \quad S = \Delta y/\Delta x = (2a + 3) + \Delta x,$$

and

(9) $$m = \lim_{\Delta x \doteq 0} S = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} [(2a + 3) + \Delta x] = 2a + 3.$$

Therefore the tangent at (a, b) is

(10) $$y - (a^2 + 3a - 5) = (2a + 3)(x - a), \text{ or } (2a + 3)x - y = a^2 + 5.$$

From (9) we observe that $m = 0$, when $2a + 3 = 0$, *i.e.* when $a = -3/2$. For all values greater than $-3/2$, $m = (2a + 3)$ is positive; for all values less than $-3/2$, m is negative. Hence the curve has a minimum at $(-3/2, -29/4)$ in Fig. 4, since the curve falls as we come toward this point and rises afterwards.

Example 3. Consider the curve $y = x^3 - 12x + 7$. If the value of x at any point P is a, the value of y is $a^3 - 12a + 7$. If the value of x at Q is $a + \Delta x$, the value of y at Q is $(a + \Delta x)^3 - 12(a + \Delta x) + 7$.

Hence

$$S = \frac{\Delta y}{\Delta x} = \frac{[(a + \Delta x)^3 - 12(a + \Delta x) + 7] - [a^3 - 12a + 7]}{\Delta x}$$

$$= (3a^2 + 3a\Delta x + \overline{\Delta x}^2) - 12,$$

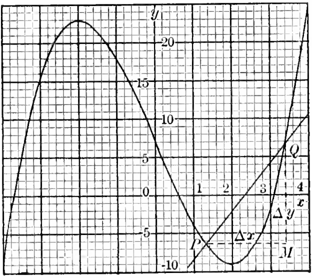


FIG. 5.

and

$$m = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = 3a^2 - 12.$$

For example, if $x = 1$, $y = -4$; at this point $(1, -4)$ the slope is $3 \cdot 1^2 - 12 = -9$; and the equation of the tangent is

$$(y + 4) = -9(x - 1), \text{ or}$$
$$9x + y - 5 = 0.$$

Since $3a^2 - 12$ is negative when $a^2 < 4$, the curve is falling when a lies between -2 and $+2$. Since $3a^2 - 12$ is positive when $a^2 > 4$, the curve is rising when $x < -2$ and when $x > +2$. At $x = \pm 2$, the slope is zero. At $x = +2$ there is a minimum (see Fig.

5), since the curve is falling before this point and rising afterwards. At $x = -2$ there is a maximum. At $x = +2$, $y = (2)^3 - 12 \cdot 2 + 7 = -9$, which is the lowest value of y near that point. At $x = -2$, $y = 23$, the highest value near it.

This information is quite useful in drawing an accurate figure. We know also that the curve rises faster and faster to the right of $x = 2$. Draw an accurate figure of your own on a large scale.

EXERCISES III. — SLOPES OF CURVES

1. Find the slope of the curve $y = x^2 + 2$ at the point where $x = 1$. Find the equation of the tangent at that point. Verify the fact that the equation obtained is a straight line, that it has the correct slope, and that it passes through the point (1, 3).

2. Draw the curve $y = x^2 + 2$ on a large scale. Through the point (1, 3) draw secants which make $\Delta x = 1, \frac{1}{2}, 0.1, 0.01$, respectively. Calculate the slope of each of these secants and show that the values are approaching the value of the slope of the curve at (1, 3).

3. Find the slope of the curve and the equation of the tangent to each of the following curves at the point mentioned. Verify each answer as in Ex. 1.

(*a*) $y = 3x^2$; (1, 3).
(*b*) $y = 2x^2 - 5$; (2, 3).
(*c*) $y = x^3$; (1, 1).
(*d*) $y = x^2 + 4x - 5$; (1, 0)
(*e*) $y = x^3 + x^2$; (1, 2).
(*f*) $y = x^3 - 3x + 4$; (2, 6).

4. Find the slope of the curve $y = x^2 - 3x + 1$ at any point $x = a$; from this find the highest (maximum) or lowest (minimum) point (if any), and show in what portions the curve is rising or falling.

5. Draw the following curves, using for greater accuracy the precise values of x and y at the highest (maximum) and the lowest (minimum) points, and the knowledge of the values of x for which the curve rises or falls. The slope of the curve at the point where $x = 0$ is also useful in (*b*), (*c*), (*e*), (*g*).

(*a*) $y = x^2 + 5x + 2$.
(*b*) $y = x^3$.
(*c*) $y = x^3 - 3x + 4$.
(*d*) $y = x^4$.
(*e*) $y = -x^2 + 3x$.
(*f*) $y = 3 + 12x - x^3$.
(*g*) $y = 2x^3 - 8x$.
(*h*) $y = x^3 - 6x + 5$.
(*i*) $y = x^3 + x^2$.

6. Show that the slope of the graph of $y = ax + b$ is always $m = a$, (1) geometrically, (2) by the methods of § 6.

7. Show that the lowest point on $y = x^2 + px + q$ is the point where $x = -p/2$, (1) by Analytic Geometry, (2) by the methods of § 6.

8. The **normal** to a curve at a point is defined in Analytic Geometry

to be the perpendicular to the tangent at that point. Its slope n is shown to be the negative reciprocal of the slope m of the tangent: $n = -1/m$. Find the slope of the normal, and the equation of the normal in Ex. 1; in each of the equations under Ex. 3.

9. The slope m of the curve $y = x^2$ at any point where $x = a$ is $m = 2a$. Show that the slope is $+1$ at the point where $a = 1/2$. Find the points where the slope has the value -1, 2, 10. Note that if the curve is drawn by taking different scales on the two axes, the slope no longer means the tangent of the angle made with the horizontal axis.

10. Find the points on the following curves where the slope has the values assigned to it;

(a) $y = x^2 - 3x + 6$; $(m = 1, -1, 2)$.
(b) $y = x^3$; $(m = 0, +1, +6)$.
(c) $y = x^3 - 3x + 4$; $(m = 9, 1)$.

11. Show that the curve $y = x^3 - 0.03x + 2$ has a minimum at $(0.1, 1.998)$ and a maximum at $(-0.1, 2.002)$. Draw the curve near the point $(0, 2)$ on a very large scale.

12. Draw each of the following curves on an appropriate scale; in each case show that the peculiar twist of the curve through its maximum and minimum would have been overlooked in ordinary plotting by points:

(a) $y = 48x^3 - x + 1$.

[Hint. Use a very small vertical scale and a rather large horizontal scale. The slope at $x = 0$ is also useful.]

(b) $y = x^3 - 30x^2 + 297x$.

[Hint. Use an exceedingly small vertical scale and a moderate horizontal scale. The slope at $x = 10$ is also useful.]

7. Speed. An important case of rate of change of a quantity is the rate at which a body moves, — its speed.

Consider the motion of a body falling from rest under the influence of gravity. During the first second it passes over 16 ft., during the next it passes over 48 ft., during the third over 80 ft. In general, if t is the number of seconds, and s the entire distance it has fallen, $s = 16t^2$ if the gravitational constant g be taken as 32. The graph of this equation (see Fig. 6) is a parabola with its vertex at the origin.

The speed, that is the rate of increase of the space passed over, is the slope of this curve, *i.e.*

$$\lim_{\Delta t \doteq 0} \Delta s/\Delta t.$$

This may be seen directly in another way. The *average* speed for an interval of time Δt is found by dividing the difference between the space passed over at the beginning and at the end of that interval of time by the difference in time: *i.e.* the average speed is the difference quotient $\Delta s \div \Delta t$. By the **speed at a given instant** we mean *the limit of the average speed* over an interval Δt beginning or ending at that instant as that interval approaches zero, *i.e.*

$$\textbf{speed} = \lim_{\Delta t \doteq 0} \Delta s/\Delta t.$$

Taking the equation $s = 16\,t^2$, if $t = 1/2$, $s = 4$ (see point P in Fig. 6). After a lapse of time Δt, the new values are $t = 1/2 + \Delta t$, and $s = 16\,(1/2 + \Delta t)^2$ (Q in Fig. 6). Then

$$\begin{aligned}\Delta s &= 16(1/2 + \Delta t)^2 - 4\\ &= 16\,\Delta t + 16\,\overline{\Delta t}^2,\end{aligned}$$

$$\Delta s/\Delta t = 16 + 16\,\Delta t.$$

Whence

$$\begin{aligned}speed &= \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t}\\ &= \lim_{\Delta t \doteq 0} (16 + 16\,\Delta t) = 16\,;\end{aligned}$$

that is, the speed at the end of the first half second is 16 ft. per second.

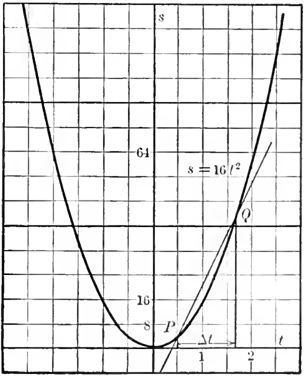


FIG. 6.

Likewise, for any value of t, say $t = T$, $s = 16\,T^2$; while for $t = T + \Delta t$, $s = 16\,(T + \Delta t)^2$; hence

$$average\ speed = \frac{\Delta s}{\Delta t} = \frac{16(T + \Delta t)^2 - 16\,T^2}{\Delta t} = 32\,T + 16\,\Delta t$$

and

$$speed = \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = 32\,T.$$

Thus, at the end of two seconds, $T = 2$, and the speed is $32 \cdot 2 = 64$, in feet per second.

8. Component Speeds. Any curve may be regarded as the path of a moving point. If a point P does move along a curve, both x and y are fixed when the time t is fixed. To specify the motion completely, we need equations which give the values of x and y in terms of t.

The *horizontal speed* is the rate of increase of x with respect to the time. This may be thought of as the speed of the projection M of P on the x-axis. As shown in § 7, this speed is the limit of the difference quotient $\Delta x \div \Delta t$ as $\Delta t \doteq 0$.

Likewise the *vertical speed* is the limit of the difference quotient $\Delta y \div \Delta t$ as $\Delta t \doteq 0$. Since the slope m of the curve P is the limit of $\Delta y \div \Delta x$ as $\Delta x \doteq 0$; and since

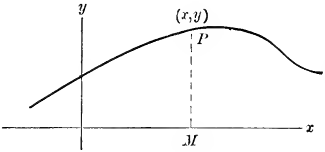


FIG. 7.

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta t} \div \frac{\Delta x}{\Delta t},$$

it follows that

$$m = (\textit{vertical speed}) \div (\textit{horizontal speed});$$

that is, the slope of the curve is the ratio of the rate of increase of y to the rate of increase of x.

9. Continuous Functions. In §§ 4–8, we have supposed that the curves used were *smooth*. The functions which we have had have all been representable by smooth curves; except perhaps at isolated points, to a small change in the value of one coördinate, there has been a correspondingly small change in the value of the other coördinate. Throughout this text, unless the contrary is expressly stated, the functions dealt with will be of the same sort. Such functions are called **continuous**. (See § 10, p. 17.)

The curve $y = 1/x$ is continuous except at the point $x = 0$; $y = \tan x$ is continuous except at the points $x = \pm \pi/2, \pm 3\pi/2$, etc. Such exceptional points occur frequently; we do not discard a curve because of them, but it is understood that any of our results may fail at such points.

EXERCISES IV. — SPEED

1. From the formula $s = 16\,t^2$, calculate the values of s when $t = 1, 2$, 1.1, 1.01, 1.001. From these values calculate the average speed between $t = 1$ and $t = 2$; between $t = 1$ and $t = 1.1$; between $t = 1$ and $t = 1.01$; between $t = 1$ and $t = 1.001$. Show that these average speeds are successively nearer to the speed at the instant $t = 1$.

2. Calculate as in Ex. 1 the average speed for smaller and smaller intervals of time after $t = 2$; and show that these approach the speed at the instant $t = 2$.

3. A body thrown vertically downwards from any height with an original velocity of 100 ft. per second, passes over in time t (in seconds) a distance s (in feet) given by the equation $s = 100\,t + 16\,t^2$ (if $g = 32$, as in § 7). Find the speed v at the time $t = 1$; at the time $t = 2$; at the time $t = 4$; at the time $t = T$.

4. In Ex. 3 calculate the average speeds for smaller and smaller intervals of time after $t = 0$; and show that they approach the original speed $v_0 = 100$. Repeat the calculations for intervals beginning with $t=2$.

5. Calculate the speed of a body at the times indicated in the following possible relations between s and t:

(a) $s = t^2$; $t = 1, 2, 10, T$. (c) $s = -16\,t^2 + 160\,t$; $t = 0, 2, 5$.

(b) $s = 16\,t^2 - 100\,t$; $t = 0, 2, T$. (d) $s = t^3 - 3\,t + 4$; $t = 0, 1/2, 1$.

6. The relation (c) in Ex. 5 holds (approximately, since $g = 32$ approximately) for a body thrown upward with an initial speed of 160 ft. per second, where s means the distance from the starting point counted positive upwards. Draw a graph which represents this relation between the values of s and t.

In this graph mark the greatest value of s. What is the value of v at that point? Find *exact* values of s and t for this point.

7. A body thrown horizontally with an original speed of 4 ft. per second falls in a vertical plane curved path so that the values of its horizontal and its vertical distances from its original position are respectively, $x = 4\,t$, $y = 16\,t^2$, where y is measured downwards. Show that the vertical speed is $32\,T$, and that the horizontal speed is 4, at the instant $t = T$. Eliminate t to show that the path is the curve $y = x^2$.

8. Show by Ex. 7 and § 8 that the slope of the curve $y = x^2$ at the point where $t = 1$, *i.e.* $(4, 16)$, is $32 \div 4$, or 8. Write the equation of the tangent at that point.

9. Show that the slope of the curve $y = x^2$ (Ex. 7) at the point (a, a^2), *i.e.* $t = a/4$, is $2a$, from Ex. 7 and § 8; and also directly by means of § 6.

10. If a body moves so that its horizontal and its vertical distances from the starting point are, respectively, $x = 16t^2$, $y = 4t$, show that its path is the curve $y^2 = x$; that its horizontal speed and its vertical speed are, respectively, $32T$ and 4, at the instant $t = T$.

11. From Ex. 10 and § 8 show that the slope of the curve $y^2 = x$ at the point $(16, 4)$, *i.e.* when $t = 1$, is $4 \div 32 = 1/8$. Write the equation of the tangent at that point.

12. From Ex. 10 and § 8 show that the slope of the curve $y^2 = x$ at the point where $t = T$ is $4 \div (32T) = 1/(8T) = 1/(2k)$, where k is the value of y at the point. Compare this result with that of Ex. 8.

10. Limits. Infinitesimals. We have been led in what precedes to make use of limits. Thus the tangent to a curve at the point P is defined by saying that its slope is the *limit* of the slope of a variable secant through P; the speed at a given instant is the *limit* of the average speed; the difference of the two values of x, Δx, was thought of as *approaching zero;* and so on. To make these concepts clear, the following precise statements are necessary and desirable.

When the difference between a variable x and a constant a becomes and remains less, in absolute value, than any preassigned positive quantity, however small, then a is the* **limit** *of the variable x.*

We also use the expression "x approaches a as a limit," or, more simply, "x approaches a." The symbol for limit is *lim*; the symbol for approaches is $\doteq$: thus we may write $\lim x = a$, or $x \doteq a$, or $\lim (a - x) = 0$, or $a - x \doteq 0$.

When the limit of a variable is *zero*, the variable is called

* When dealing with real numbers, absolute value is the value without regard to signs so that the absolute value of -2 is 2. A convenient symbol for it is two vertical lines; thus $|3 - 7| = 4$.

an **infinitesimal.** Thus $a - x$ above is an infinitesimal. The difference between any variable and its limit is always an infinitesimal. *When a variable x approaches a limit a, any* **continuous** *function $f(x)$ approaches the limit $f(a)$*: thus, if $y=f(x)$ and $b=f(a)$, we may write

$$\lim_{x \doteq a} y = b, \text{ or } \lim_{x \doteq a} f(x) = f(a).$$

This condition is the precise definition of continuity at the point $x = a$. (See § 9, p. 14.)

11. Properties of Limits. The following properties of limits will be assumed as self-evident; some of them have already been used in the articles noted below.

THEOREM A. *The limit of the sum of two variables is the sum of the limits of the two variables.* This is easily extended to the case of more than two variables. (Used in §§ 4, 6, and 7.)

THEOREM B. *The limit of the product of two variables is the product of the limits of the variables.* (Used in §§ 4, 6, and 7.)

THEOREM C. *The limit of the quotient of one variable divided by another is the quotient of the limits of the variables,* **provided** *the limit of the divisor is not zero.* (Used in § 8.)

The exceptional case in Theorem C is really the most interesting and important case of all. The exception arises because when zero occurs as a denominator, the division cannot be performed. In finding the slope of a curve, we consider $\lim(\Delta y/\Delta x)$ *as Δx approaches zero;* notice that this is precisely the case ruled out in Theorem C. Again, the speed is $\lim(\Delta s/\Delta t)$ *as Δt approaches zero.* The limit of any such *difference quotient* is one of these exceptional cases.

Now it is clear that the slope of a curve (or the speed of an object) may have a great variety of values in different cases: *no one answer is sufficient* for all examples, in the case of the limit of a quotient when the denominator approaches zero.

THEOREM D. *The limit of the ratio of two infinitesimals depends upon the law connecting them; otherwise it is quite indeterminate.* Of this the student will see many instances; for *the Differential Calculus consists of the consideration of just such limits.* In fact, the very reason for the existence of the Differential Calculus is that the exceptional case of Theorem C is important, and cannot be settled in an offhand manner.

The thing to be noted here is, that, no matter how small two quantities may be, their ratio may be either small or large; and that, if the two quantities are variables whose limit is zero, the limit of their ratio may be either finite, zero, or non-existent. In our work with such forms we shall try to substitute an equivalent form whose limit can be found. Obviously, to say that two variables are vanishing implies nothing about the limit of their *ratio.*

12. Ratio of an Arc to its Chord. Another important illustration of a ratio of infinitesimals is the ratio of the chord of a curve to its subtended arc:

$$R = \frac{\text{chord } PQ}{\text{arc } PQ}.$$

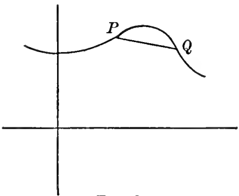


FIG. 8.

If Q approaches P, both the arc and the chord approach zero. At any stage of the process the arc is greater than the chord; but as Q approaches P this difference diminishes very rapidly, and the ratio R approaches 1:

$$\lim_{Q \doteq P} R = \lim_{PQ \doteq 0} \frac{\text{chord } PQ}{\text{arc } PQ} = 1.$$

This property is self-evident because it amounts to the same thing as the **definition** of the length of the curve; we ordinarily think of the length of an arc of a curve as the limit of the length of an inscribed broken line, as the lengths of the

segments of the broken line approach zero. Thus, the length of circumference of a circle is defined to be the limit of the perimeter of an inscribed polygon as the lengths of all its sides approach zero. This would not be true if the ratio of an arc to its chord did not approach 1.*

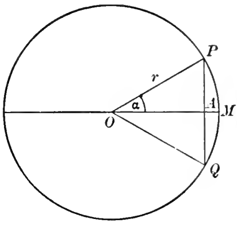


Fig. 9.

13. Ratio of the Sine of an Angle to the Angle. In a circle, the arc PQ and the chord PQ can be expressed in terms of the angle at the center. Let $\alpha = \angle QOP/2$; then arc $PQ = 2\,\alpha \times r$ if α is measured in *circular measure* (see *Tables*, II, F, 3); and the chord $PQ = 2\,r \sin \alpha$, since $r \sin \alpha = AP$.

It follows that

$$\lim_{\alpha \doteq 0} \frac{\text{chord } PQ}{\text{arc } PQ} = \lim_{\alpha \doteq 0} \frac{2\,r \sin \alpha}{2\,r\alpha} = \lim_{\alpha \doteq 0} \frac{\sin \alpha}{\alpha} = \mathbf{1};$$

hence

$$\lim_{\alpha \doteq 0} \frac{\sin \alpha}{\alpha} = 1,$$

for we have just seen that the limit of the ratio of an infinitesimal chord to its arc is 1.

This result is very important in later work; just here it serves as a new illustration of the ratio of infinitesimals: *the ratio of the sine of an angle to the angle itself (measured in circular measure) approaches 1 as the angle approaches zero.*

14. Infinity. Theorem D accounts for the case when the numerator as well as the denominator in Theorem C is infinitesimal. There remains the case when the denominator only

* This point of view is fundamental. See Goursat-Hedrick, *Mathematical Analysis*, Vol. I, § 80, p. 161. At some exceptional points the property may fail, but such points we always subject to special investigation.

is infinitesimal. *A variable whose reciprocal is infinitesimal is said to* **become infinite** *as the reciprocal approaches zero.*

Thus $y=1/x$ is a variable whose reciprocal is x. As x approaches zero, y is said to become infinite. Notice however that y has no value whatever when $x=0$.

Likewise $y=\sec x$ is a variable whose reciprocal, $\cos x$, is infinitesimal as x approaches $\pi/2$; hence we say that $\sec x$ becomes infinite as x approaches $\pi/2$.

In any case, it is clear that a variable which becomes infinite becomes and remains larger in absolute value than any preassigned positive number, however large.

The student should carefully notice that infinity is not a number; when we say that "$\sec x$ becomes infinite as x approaches $\pi/2$," * we do not mean that $\sec(\pi/2)$ has a value, we merely tell what occurs when x approaches $\pi/2$.

EXERCISES V. — LIMITS AND INFINITESIMALS

1. Imagine a point traversing a line-segment in such fashion that it traverses half the segment in the first second, half the remainder in the next second, and so on; always half the remainder in the next following second. Will it ever traverse the entire line? Show that the remainder after t seconds is $1/2^t$, if the total length of the segment is 1. Is this infinitesmal? Why?

2. Show that the distance traversed by the point in Ex. 1 in t seconds is $1/2+1/2^2+\cdots+1/2^t$. Show that this sum is equal to $1-1/2^t$; hence show that its limit is 1. Show that in any case the limit of the distance traversed is the total distance, as t increases indefinitely.

3. Show that the limit of $3-x^2$ as x approaches zero is 3. State this result in the symbols used in § 10. Draw the graph of $y=3-x^2$ and show that y approaches 3 as x approaches zero.

4. Evaluate the following limits:

(*a*) $\lim\limits_{x\doteq 0}(2-5x+3x^2)$. (*d*) $\lim\limits_{x\doteq 1}\dfrac{3-2x^2}{4+2x^2}$. (*g*) $\lim\limits_{x\doteq 7}\dfrac{x^2-3x+2}{x^2+2x+3}$.

(*b*) $\lim\limits_{x\doteq 1}(2-5x+3x^2)$. (*e*) $\lim\limits_{x\doteq 0}\dfrac{x}{1-x}$. (*h*) $\lim\limits_{x\doteq 0}\dfrac{a+bx}{c+dx}$.

(*c*) $\lim\limits_{x\doteq k}(a+bx+cx^2)$. (*f*) $\lim\limits_{x\doteq 1}\dfrac{1-x}{x}$. (*i*) $\lim\limits_{x\doteq 0}\dfrac{a+bx+cx^2}{m+nx+lx^2}$.

* Or, as is stated in short form in many texts, "$\sec(\pi/2)=\infty$."

5. If the numerator and denominator of a fraction contain a common factor, that factor may be canceled in finding a limit, since the value of the fraction which we use is not changed. Evaluate before and after canceling a common factor:

(a) $\lim\limits_{x \doteq 1} \dfrac{(x+2)(x+1)}{(2x+3)(x+1)}$. (b) $\lim\limits_{x \doteq 0} \dfrac{x(x+2)}{(x+1)(x+2)}$.

Evaluate after (not before) removing a common factor:

(c) $\lim\limits_{x \doteq 0} \dfrac{x^2}{x}$. (d) $\lim\limits_{x \doteq 1} \dfrac{x^2-3x+2}{x^2-1}$. (e) $\lim\limits_{x \doteq 1} \dfrac{(x+2)(x-1)}{(2x+3)(x-1)}$.

(f) $\lim\limits_{x \doteq 1} \dfrac{\sqrt{x}-1}{x-1}$. (g) $\lim\limits_{x \doteq 0} \dfrac{x^2(x+1)^2}{x^3+2x^2}$. (h) $\lim\limits_{x \doteq 0} \dfrac{x^n}{x} = \begin{cases} 0, & n>1, \\ 1, & n=1. \end{cases}$

6. Show that

$$\lim_{x \doteq \infty} \frac{2x^2+3}{x^2+4x+5} = 2.$$

[Hint. Divide numerator and denominator by x^2; then such terms as $3/x^2$ approach zero as x becomes infinite.]

7. Evaluate:

(a) $\lim\limits_{x \doteq \infty} \dfrac{2x+1}{3x+2}$. (b) $\lim\limits_{x \doteq \infty} \dfrac{2x^2-4}{3x^2+2}$. (c) $\lim\limits_{x \doteq \infty} \dfrac{ax+b}{mx+n}$.

(d) $\lim\limits_{x \doteq \infty} \dfrac{x}{\sqrt{1+x^2}}$. (e) $\lim\limits_{x \doteq \infty} \dfrac{\sqrt{1+x^2}}{\sqrt{x^2-1}}$. (f) $\lim\limits_{x \doteq \infty} \dfrac{\sqrt{ax^2+bx+c}}{mx+n}$.

8. Let O be the center of a circle of radius $r = OB$, and let $\alpha = \angle COB$ be an angle at the center. Let BT be perpendicular to OB, and let BF be perpendicular to OC. Show that OF approaches OC as α approaches zero; likewise arc $CB \doteq 0$, arc $DB \doteq 0$, and $FC \doteq 0$, as $\alpha \doteq 0$.

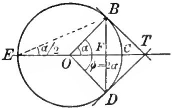


Fig. 10.

9. In the figure of Ex. 8, show that the obvious geometric inequality $FB <$ arc $CB < BT$ is equivalent to $r \sin \alpha < r \cdot \alpha < r \tan \alpha$, if α is measured in circular measure. Hence show that $\alpha/\sin \alpha$ lies between 1 and $1/\cos \alpha$, and therefore that $\lim (\alpha/\sin \alpha) = 1$ as $\alpha \doteq 0$. (Verification of § 13.)

10. In the figure of Ex. 8, show that

$$\lim_{\alpha \doteq 0} \frac{FB}{r} = 0; \quad \lim_{\alpha \doteq 0} \frac{OF}{r} = 1; \quad \lim_{\alpha \doteq 0} \frac{BT}{r} = 0; \quad \lim_{\alpha \doteq 0} \frac{FC}{r} = 0; \quad \lim_{\alpha \doteq 0} \frac{\text{arc } CB}{r} = 0.$$

11. Show that the following quantities become infinite as the independent variable approaches the value specified; in (*a*) and (*b*) draw the graph.

(*a*) $\lim\limits_{x \doteq 0} \dfrac{1}{x^2}$. (*c*) $\lim\limits_{a \doteq 0} \dfrac{r}{FC}$, (Ex. 8). (*e*) $\lim\limits_{x \doteq 0} \dfrac{x^n}{x}$, $(n < 1)$.

(*b*) $\lim\limits_{x \doteq 1} \dfrac{x+2}{x-1}$. (*d*) $\lim\limits_{a \doteq 0} \dfrac{r}{BT}$, (Ex. 8). (*f*) $\lim\limits_{x \doteq 2} \dfrac{2x+3}{\sqrt{x^2-3x+2}}$.

12. As the chord of a circle approaches zero, which of the following ratios has a finite limit, which is infinitesimal, and which is becoming infinite: the chord to its arc; the radius to the chord; the sector of the arc to the triangle cut off by the chord; the area of the circle to the sector; the chord of twice the arc to the chord of thrice the arc; the radius of the circle to the chord of an arc a thousand times the given arc?

13. Is the sum of two infinitesimals itself infinitesimal? Is the difference? Is the product? Is the quotient? Is a constant times an infinitesimal an infinitesimal?

14. If to each of two integers an infinitesimal is added, show that the ratio of these sums differs from that of the integers by an infinitesimal. [See Ex. 4 (*h*).]

15. Show that the graph of $y = f(x)$ has a vertical asymptote if $f(x)$ becomes infinite as $x \doteq a$. Illustrate this by drawing the following graphs:

(*a*) $y = \dfrac{3x+2}{x-2}$. (*c*) $y = \dfrac{1}{1-\cos x}$. (*e*) $y = \dfrac{1}{\sqrt{1-x^2}}$.

(*b*) $y = \dfrac{2x-1}{(x+1)(x-5)}$. (*d*) $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$. (*f*) $y = \dfrac{ax+b}{cx+d}$.

15. Derivatives. While such illustrations as those in § 12 and Exercises V, above, are interesting and reasonably important, by far the most important cases of the ratio of two infinitesimals are those of the type studied in §§ 4–8, in which each of the infinitesimals is the difference of two values of a variable, such as $\Delta y/\Delta x$ or $\Delta s/\Delta t$. Such a difference quotient $\Delta y/\Delta x$ of y with respect to x evidently represents the **average** rate of increase of y with respect to x in the interval Δx; if x represents time and y distance, then $\Delta y/\Delta x$ is the average speed over the interval Δx (§ 7, p. 13); if $y = f(x)$ is thought

of as a curve, then $\Delta y/\Delta x$ is the slope of a secant or the average rate of rise of the curve in the interval Δx (§ 4, p. 6).

The *limit* obtained in such cases represents the **instantaneous** *rate of increase* of one variable with respect to the other, — this may be the slope of a curve, or the speed of a moving object, or some other *rate*, depending upon the nature of the problem in which it arises.

In general, *the limit of the quotient* $\Delta y/\Delta x$ *of two infinitesimal differences is called* **the derivative of y with respect to x**; *it is represented by the symbol* dy/dx:

$$\frac{dy}{dx} \equiv \textbf{derivative of } y \textbf{ with respect to } x = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}.$$

Henceforth we shall use this new symbol dy/dx or other convenient abbreviations; * but the student must not forget the real meaning: *slope,* in the case of curve; *speed,* in the case of motion; some other tangible concept in any new problem which we may undertake; *in every case the rate of increase of y with respect to x.*

Any mathematical formulas we obtain will apply in any of these cases; we shall use the letters x and y, the letters s and t, and other suggestive combinations; but the student should remember that any formula written in x and y also holds true, for example, with the letters s and t, or for any other pair of letters.

16. Formula for Derivatives. If we are to find the value of a derivative, as in §§ 4–7, we must have given one of the variables y as a function of the other x:

(1) $$y = f(x).$$

If we think of (1) as a curve, we may, as in § 4, take any

* Often read "*the x* derivative of y." Other names sometimes used are *differential coefficient*, and *derived function*. Other convenient notations often used are $D_x y$, y_x, y', $\dot{y}$; the last two are not safe unless it is otherwise clear what the indepeudent variable is.

point P whose coördinates are x and y, and join it by a secant PQ to any other point Q, whose coördinates are $x+\Delta x$, $y+\Delta y$.

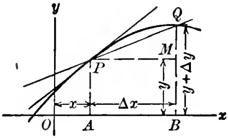


FIG. 11.

Here x and y represent fixed values of x and y; this will prove more convenient than to use new letters each time, as we did in §§ 4–7.

Since P lies on the curve (1), its coördinates (x, y) satisfy the equation (1), $y=f(x)$. Since Q lies on (1), $x+\Delta x$ and $y+\Delta y$ satisfy the same equation; hence we must have

(2) $$y+\Delta y=f(x+\Delta x).$$

Subtracting (1) from (2) we get

(3) $$\Delta y=f(x+\Delta x)-f(x);$$

whence the *difference quotient* is

(4) $$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}=\textit{average slope over } PM,$$

and therefore the *derivative* is

(5) $$\frac{dy}{dx}=\lim_{\Delta x \doteq 0}\frac{\Delta y}{\Delta x}\equiv\lim_{\Delta x \doteq 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\textbf{slope at } P.*$$

This formula is often convenient; we shall apply it at once.

17. Rule for Differentiation. The process of finding a derivative is called **differentiation.** To apply formula (5) of § 16:

(*A*) *Find* $(y+\Delta y)$ *by substituting* $(x+\Delta x)$ *for* x *in the given function or equation;* this gives $y+\Delta y=f(x+\Delta x)$.

(*B*) *Subtract* y *from* $y+\Delta y$*;* this gives $\Delta y=f(x+\Delta x)-f(x)$.

(*C*) *Divide* Δy *by* Δx *to find the difference quotient* $\Delta y/\Delta x$*; simplify this result.*

(*D*) *Find the* **limit** *of* $\Delta y/\Delta x$ *as* Δx *approaches zero;* this result is the **derivative**, dy/dx.

* Instead of *slope*, read *speed* in case the problem deals with a motion, as in § 7. In general, $\Delta y/\Delta x$ is the *average* rate of increase, and dy/dx is the *instantaneous* rate.

Example 1. Given $y = f(x) \equiv x^2$, to find dy/dx.

(*A*) $$f(x + \Delta x) = (x + \Delta x)^2.$$

(*B*) $$\Delta y = f(x + \Delta x) - f(x) = (x + \Delta x)^2 - x^2 = 2x\Delta x + \overline{\Delta x}^2.$$

(*C*) $$\Delta y/\Delta x = (2x\Delta x + \overline{\Delta x}^2) \div \Delta x = 2x + \Delta x.$$

(*D*) $$dy/dx = \lim_{\Delta x \doteq 0} \Delta y/\Delta x = \lim_{\Delta x \doteq 0} (2x + \Delta x) = 2x.$$

Compare this work and the answer with the work of § 4, p. 6.

Example 2. Given $y = f(x) \equiv x^3 - 12x + 7$, to find dy/dx.

(*A*) $$f(x + \Delta x) = (x + \Delta x)^3 - 12(x + \Delta x) + 7.$$

(*B*) $$\Delta y = f(x + \Delta x) - f(x) = 3x^2\Delta x + 3x\overline{\Delta x}^2 + \overline{\Delta x}^3 - 12\Delta x.$$

(*C*) $$\Delta y/\Delta x = 3x^2 + 3x\Delta x + \overline{\Delta x}^2 - 12.$$

(*D*) $$dy/dx = \lim_{\Delta x \doteq 0} \Delta y/\Delta x = \lim_{\Delta x \doteq 0} (3x^2 + 3x\Delta x + \overline{\Delta x}^2 - 12) = 3x^2 - 12.$$

Compare this work and the answer with the work of Example 3, § 6.

Example 3. Given $y = f(x) \equiv 1/x^2$, to find dy/dx.

(*A*) $$f(x + \Delta x) = \frac{1}{(x + \Delta x)^2}.$$

(*B*) $$\Delta y = f(x + \Delta x) - f(x) = \frac{1}{(x + \Delta x)^2} - \frac{1}{x^2} = -\frac{2x\Delta x + \overline{\Delta x}^2}{x^2(x + \Delta x)^2}.$$

(*C*) $$\Delta y/\Delta x = -\frac{2x + \Delta x}{x^2(x + \Delta x)^2}.$$

(*D*) $$dy/dx = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \left[-\frac{2x + \Delta x}{x^2(x + \Delta x)^2}\right] = -\frac{2x}{x^4} = -\frac{2}{x^3}.$$

Example 4. Given $y = f(x) \equiv \sqrt{x}$, to find dy/dx, or $df(x)/dx$.

(*A*) $$f(x + \Delta x) = \sqrt{x + \Delta x}.$$

(*B*) $$\Delta y = f(x + \Delta x) - f(x) = \sqrt{x + \Delta x} - \sqrt{x}.$$

(*C*) $$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$
$$= \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}.$$

(*D*) $$\frac{dy}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

(Compare Ex. 11, p. 16.)

Example 5. Given $y = f(x) \equiv x^7$, to find $df(x)/dx$.

(*A*) $f(x + \Delta x) = (x + \Delta x)^7 = x^7 + 7x^6\Delta x +$ (terms with a factor $\overline{\Delta x}^2$).

(*B*) $\Delta y = f(x + \Delta x) - f(x) = 7x^6\Delta x +$ (terms with a factor $\overline{\Delta x}^2$).

(*C*) $\Delta y/\Delta x = 7x^6 +$ (terms with a factor Δx).

(*D*) $dy/dx = \lim_{\Delta x \doteq 0} \Delta y/\Delta x = \lim_{\Delta x \doteq 0} [7x^6 + \text{(terms with a factor } \Delta x)] = 7x^6$.

EXERCISES VI. — FORMAL DIFFERENTIATION

1. Find the derivative of $y = x^3$ with respect to x. [Compare Ex. 3 (*c*), p. 11.] Write the equation of the tangent at the point (2, 8) to the curve $y = x^3$.

2. Find the derivatives of the following functions with respect to x:

(*a*) $x^2 - 3x + 4$. (*b*) $x^3 - 6x + 7$. (*c*) $x^4 + 5$.

(*d*) $x^4 + 3x^2 - 2$. (*e*) $x^3 + 2x^2 - 4$. (*f*) $x^4 - 3x^3 + 5x$.

(*g*) $\dfrac{1}{x^2}$. (*h*) $\dfrac{1}{x+1}$. (*i*) $\dfrac{1}{2x-3}$.

(*j*) $\sqrt{x+1}$. (*k*) $\dfrac{x}{x+1}$. (*l*) $\dfrac{2x+3}{x-2}$.

3. Find the equation of the tangent and the equation of the normal to the curve $y = 1/x$ at the point where $x = 2$. (See Ex. 8, p. 11.)

4. Find the values of x for which the curve $y = x^3 - 15x + 1$ rises and those for which it falls; find the highest point (maximum) and the lowest point (minimum). Draw the graph accurately.

5. Draw accurate graphs for the following curves:

(*a*) $y = x^3 - 18x + 3$. (*c*) $y = x^4 - 32x$.

(*b*) $y = x^3 + 3x^2$. (*d*) $y = x^4 - 18x^2$.

6. Determine the speed of a body which moves so that

$$s = 16t^2 + 10t + 5.$$

[A body thrown down from a height with initial speed 10 ft. per second moves in this way approximately, if s is measured downward from a mark 5 ft. above the starting point.]

7. If a body moves so that its horizontal and its vertical distances from a point are, respectively, $x = 10t$, $y = -16t^2 + 10t$, find its horizontal speed and its vertical speed. Show that the path is

$$y = -16x^2/100 + x,$$

and that the slope of this path is the ratio of the vertical speed to the

horizontal speed. [These equations represent, approximately, the motion of an object thrown upward at an angle of 45° with a speed $10\sqrt{2}$.]

8. A stone is dropped into still water. The circumference c of the growing circular waves thus made, as a function of the radius r, is $c = 2\pi r$.

Show that $dc/dr = 2\pi$, *i.e.* that the circumference changes 2π times as fast as the radius.

Let A be the area of the circle. Show that $dA/dr = 2\pi r$; *i.e.* the rate at which the area is changing compared to the radius is numerically equal to the circumference.

9. Determine the rates of change of the following variables:

(*a*) The surface of a sphere compared with its radius, as the sphere expands.

(*b*) The volume of a cube compared with its edge, as the cube enlarges.

(*c*) The volume of a right circular cone compared with the radius of its base (the height being fixed), as the base spreads out.

10. If a man 6 ft. tall is at a distance x from the base of an arc light 10 ft. high, and if the length of his shadow is s, show that $s/6 = x/4$, or $s = 3x/2$. Find the rate (ds/dx) at which the length s of his shadow increases as compared with his distance x from the lamp base.

11. The *specific heat* of a substance (*e.g.* water) is the amount of heat required to raise the temperature of a unit volume of that substance 1° (Centigrade). This amount is known to change for the same substance for different temperatures. The average specific heat between two temperatures is the ratio of the quantity of heat ΔH consumed in raising the temperature divided by the change Δt in the temperature; show that the actual specific heat at a given temperature is dH/dt.

12. The *coefficient of expansion* of a solid substance is the amount a bar of that substance 1 ft. long will expand when the temperature changes 1°. Express the average coefficient of expansion, and show that the coefficient of expansion at any given temperature is dl/dt, if the bar is precisely 1 ft. long at that temperature. (See also Ex. 12, p. 145.)

CHAPTER III

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

PART I. EXPLICIT FUNCTIONS

18. Classification of Functions. For convenience it is usual to classify functions into certain groups.

A function which can be expressed directly in terms of the independent variable x by means of the three elementary operations of multiplication, addition, and subtraction is called a **polynomial** in x.

Thus, $x^2(=x \cdot x)$, $2x^3+4x^2-7x+3$, x^3-4x+6, etc., are polynomials. The most general polynomial is $a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$, where the coefficients $a_0, a_1, \cdots, a_n$ are constants, and the exponents are positive integers. Notice that raising a quantity to a positive integral power can be regarded as a succession of multiplications.

A function which can be expressed directly in terms of the independent variable x by means of the four elementary operations of multiplication, division, addition, and subtraction, is called a **rational function** of x. Thus, $1/x$, $(x^3-3x)/(2x+7)$, etc., are rational. The most general rational function is the quotient of two polynomials, since more than one division can be reduced to a single division by the rules for the combination of fractions. All polynomials are also rational functions.

If, besides the four elementary operations, a function requires for its direct expression in the independent variable x at most the extraction of integral roots, it is called a **simple algebraic function** * of x. Thus, $\sqrt{x}$, $(\sqrt{x^2+1}-2)/(3-\sqrt[3]{x})$,

* Since the expression "algebraic function" is used in the broader sense of § 27 in advanced mathematics, we shall call these *simple* algebraic functions.

etc., are simple algebraic functions. All rational functions are also simple algebraic functions.

Simple algebraic functions which are not rational are called **irrational functions.**

A function which is not an algebraic function is called a **transcendental function.** Thus, $\sin x$, $\log x$, e^x, $x^2 + \tan^{-1}(1+x)$, etc., are transcendental.

In this chapter we shall deal only with algebraic functions.

19. Differentiation of Polynomials. We have differentiated a number of polynomials in Chapter II. To simplify the work to a mere matter of routine, we need four rules:

The derivative of a constant is zero:

$$\text{[I]} \qquad \frac{dc}{dx} = 0.$$

The derivative of a constant times a function is equal to the constant times the derivative of the function:

$$\text{[II]} \qquad \frac{d(c \cdot u)}{dx} = c \cdot \frac{du}{dx}.$$

The derivative of the sum of two functions is equal to the sum of their derivatives:

$$\text{[III]} \qquad \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

The derivative of a power, x^n, with respect to x is nx^{n-1}:

$$\text{[IV]} \qquad \frac{dx^n}{dx} = nx^{n-1}.$$

[We shall prove this at once in the case when n is a positive integer; later we shall prove that it is true also for negative and fractional values of n.]

Each of these rules was illustrated in Chapter II, § 17. To prove them we use the rule of § 17.

Proof of [I]. If $y = c$, a change in x produces no change in y; hence $\Delta y = 0$. Therefore $dy/dx = \lim \Delta y/\Delta x = \lim 0 = 0$ as Δx approaches zero. Geometrically, the slope of the curve $y = c$ (a horizontal straight line) is everywhere zero.

Proof of [II]. If $y = c \cdot u$ where u is a function of x, a change Δx in x produces a change Δu in u and a change Δy in y; following the rule of § 17 we find:

$$(A) \qquad y + \Delta y = c \cdot (u + \Delta u).$$

$$(B) \qquad \Delta y = c \cdot \Delta u.$$

$$(C) \qquad \Delta y/\Delta x = c \cdot (\Delta u/\Delta x).$$

$$(D) \qquad dy/dx = \lim_{\Delta x \doteq 0} [c \cdot (\Delta u/\Delta x)] = c \cdot \lim_{\Delta x \doteq 0} \Delta u/\Delta x = c(du/dx).$$

Thus $d(7\,x^2)/dx = 7 \cdot d(x^2)/dx = 7 \cdot 2\,x = 14\,x$. (See §§ 4, 17.)

Proof of [III]. If $y = u + v$, where u and v are functions of x, a change Δx in x produces changes Δy, Δu, Δv in y, u, v, respectively, hence

$$(A) \qquad y + \Delta y = (u + \Delta u) + (v + \Delta v);$$

$$(B) \qquad \Delta y = \Delta u + \Delta v;$$

$$(C) \qquad \Delta y/\Delta x = \Delta u/\Delta x + \Delta v/\Delta x;$$

$$(D) \qquad dy/dx = \lim_{\Delta x \doteq 0} (\Delta u/\Delta x) + \lim_{\Delta x \doteq 0} (\Delta v/\Delta x) = du/dx + dv/dx.$$

Thus

$$\frac{d(x^3 - 12\,x + 7)}{dx} = \frac{d(x^3)}{dx} - \frac{d(12\,x)}{dx} + \frac{d(7)}{dx} = 3\,x^2 - 12 + 0,$$

by applying the preceding rules and noticing that $dx^3/dx = 3\,x^2$. [See Ex. 1 of Exercises VI and compare Example 3, p. 10, and Example 2, p. 25].

Proof of [IV]. If $y = x^n$, we proceed as in Example 5, p. 26:

(A) $y + \Delta y = (x + \Delta x)^n = x^n + nx^{n-1}\Delta x +$ (terms which have a common factor $\overline{\Delta x}^2$).

(B) $\Delta y = nx^{n-1}\Delta x +$ (terms with a common factor $\overline{\Delta x}^2$).

(C) $\Delta y/\Delta x = nx^{n-1} +$ (terms which have a factor Δx).

$$(D) \qquad dy/dx = \lim_{\Delta x \doteq 0} (\Delta y/\Delta x) = nx^{n-1}.$$

This proof holds good only for positive integral values of n. For negative and fractional values of n, see §§ 20, 23.

Example 1. $d(x^9)/dx = 9\,x^8$. (This would be serious without the rule.)

Example 2. $dx/dx = 1 \,.\, x^0 = 1$, since $x^0 = 1$.

This is also evident directly: $dx/dx = \lim\limits_{\Delta x \doteq 0} \Delta x/\Delta x = 1$. Notice however that no new rule is necessary.

Example 3. $\dfrac{d}{dx}(x^4 - 7\,x^2 + 3\,x - 5) = 4\,x^3 - 14\,x + 3.$

Example 4. $\dfrac{d}{dx}(Ax^m + Bx^n + C) = mAx^{m-1} + nBx^{n-1}.$

EXERCISES VII. — DIFFERENTIATION OF POLYNOMIALS

Calculate the derivative of each of the following expressions with respect to the independent variable it contains (x or r or s or t or y or u). In this list, the first letters of the alphabet, down to n, inclusive, represent constants.

1. (*a*) $y = 5\,x^3$. (*d*) $y = 5\,(x^3 + 1)$. (*g*) $y = -10\,x^{10} + 10$.
   (*b*) $y = x^4/4$. (*e*) $y = (x^4 - 2)/4$. (*h*) $y = 8\,x^5 + 6\,x^4$.
   (*c*) $y = 5\,x^3 + 1$. (*f*) $y = -10\,x^{10}$. (*i*) $y = 7\,x^6 - 6\,x^7 + 5$.

2. (*a*) $y = ax^3$. (*c*) $y = (a + b)\,x^3$. (*e*) $y = hx^5 - kx^6 + l$.
   (*b*) $y = -c^2x^9$. (*d*) $y = (a^2 - b^2)\,x^4$. (*f*) $y = A + Bx + Cx^2$.

3. (*a*) $s = kt^2$. (*b*) $s = t^2(3\,t + t^2)$. (*c*) $s = c\,(at^3 + bt^4)$.

4. (*a*) $q = s\,(s^2 - 1)$. (*c*) $q = (1 - s^3)\,(2 + s^5)$.
   (*b*) $q = s^2\,(a - bs + cs^2)$. (*d*) $q = as\,(b + cs) + d$.

5. (*a*) $z = (y + a)(y - b)$. (*c*) $z = (y^{10} + 2)(y^{10} - 3)$.
   (*b*) $z = ay^3(y^7 + by^6)$. (*d*) $z = (3\,y^2 + 2)^2$.

6. (*a*) $v = (hu^4 - ku^2 + l)\,u^8$. (*b*) $v = a\,(u^2 + u + 1)(u^2 - u + 1)$.

7. (*a*) $y = kx^n + lx^m$. (*b*) $y = ax^{2n} - bx^{3m}$.

8. (*a*) $y = x^{2n+m} + x^{n+2m} + k$. (*b*) $y = a + bx^{n^2}$.

9. Determine the slope of the curve $y = x^2 - 2\,x$ at the origin. Where is the slope 2? Where is the tangent horizontal? Draw the graph.

10. Locate the vertex of the parabola $y = x^2 + 8\,x + 19$ by finding the point at which the tangent is horizontal.

11. Proceed for each of the following curves as in Ex. 10:
(*a*) $y = x^2 - 2\,x + 2$. (*b*) $y = -x^2 + 2\,x - 10$. (*c*) $y = ax^2 + bx + c$

12. Where on the parabola $y = x^2$ is the slope 1? Where is the slope 1 on the curve $y = x^3$? On $y = x^4$? On $y = x^n$? Where is the slope 0 on each of these curves?

13. What is the slope of the curve $y = 2x^3 - 3x^2 + 4$ at $x = 0, \pm 2, \pm 4$? Where is the slope $9/2$? $-3/2$? Where is the tangent horizontal; are these points highest or lowest points, or neither? Draw the graph.

14. What is the slope of the curve $y = x^4/4 - 2x^3 + 4x^2$ at $x = 0, 1, -1, -2$? Where is the tangent horizontal; are these points maxima or minima? Where is the slope equal to eight times the value of x.

15. Show that the function $x^3 + 3x^2 + 3x + 1$ always increases with x. Where is the tangent horizontal? Show that there is no maximum or minimum at this point.

16. Locate the maxima and minima (if any exist) on each of the following curves and draw their graphs accurately:

(*a*) $y = x^3 - 27x + 15$.

(*b*) $y = 2x^3 - 9x^2 + 12x - 10$.

(*c*) $y = x^3 - 9x^2 + 27x - 15$.

(*d*) $y = 4x^3 - 11x^2 - 70x + 20$.

(*e*) $y = 3x^4 - 4x^3 + 5$.

(*f*) $y = 3x^5 - 80x^3 + 1000$.

17. At what angle does the line $y = 2x$ meet the parabola $y = x^2 + 4x + 1$?

18. Find the angle between the curve $y = x^3$ and the straight line $y = 9x$ at each of their points of intersection.

19. At what angles does the curve $y = (x - 1)(x - 2)(x - 3)$ cut the x-axis?

20. If a sphere expands — as when a rubber balloon is distended, or when an orange is growing — the volume and the radius both increase. Find the rate of growth of the volume with respect to the radius.

21. In an expanding sphere, find the rate of growth of the surface with respect to the radius.

22. Find the rate of change of the total surface of a right circular cylinder with respect to the radius, the altitude being fixed; with respect to the altitude when the radius is fixed.

Do the same for a right circular cone.

20. Differentiation of Rational Functions. In order to differentiate all rational functions, we need only one more rule, — that for differentiating a *fraction*.

The derivative of a quotient N/D of two functions N and D is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator:

$$\text{[V]} \qquad \frac{d\left(\frac{N}{D}\right)}{dx} = \frac{D\frac{dN}{dx} - N\frac{dD}{dx}}{D^2}.$$

To prove this rule, let $y = N/D$, where N and D are functions of x; then a change Δx in x produces changes Δy, ΔN, ΔD in y, N, and D, respectively; hence, by the rule of § 17:

$$(A) \qquad y + \Delta y = \frac{N + \Delta N}{D + \Delta D};$$

$$(B) \qquad \Delta y = \frac{N + \Delta N}{D + \Delta D} - \frac{N}{D} = \frac{D \cdot \Delta N - N \cdot \Delta D}{D(D + \Delta D)};$$

$$(C) \qquad \frac{\Delta y}{\Delta x} = \frac{D\frac{\Delta N}{\Delta x} - N\frac{\Delta D}{\Delta x}}{D(D + \Delta D)};$$

$$(D) \qquad \frac{dy}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \frac{D\frac{dN}{dx} - N\frac{dD}{dx}}{D^2}.$$

Example 1. $$\frac{d}{dx}\left(\frac{x^2+3}{3x-7}\right) = \frac{(3x-7)\frac{d}{dx}(x^2+3) - (x^2+3)\frac{d}{dx}(3x-7)}{(3x-7)^2}$$
$$= \frac{(3x-7)\cdot 2x - (x^2+3)\cdot 3}{(3x-7)^2} = \frac{3x^2-14x-9}{(3x-7)^2}.$$

Example 2. $$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{x^2\frac{d1}{dx} - 1\cdot\frac{d(x^2)}{dx}}{(x^2)^2} = \frac{0-2x}{x^4} = -\frac{2}{x^3}.$$

(Compare Example 3, § 17, p. 25.)

Example 3. $$\frac{d}{dx}(x^{-k}) = \frac{d}{dx}\left(\frac{1}{x^k}\right) = \frac{0 - kx^{k-1}}{x^{2k}} = \frac{-k}{x^{k+1}} = -kx^{-k-1}.$$

Note that formula IV holds also when n is a negative integer, for if $n = -k$, formula IV gives the result we have just proved.

D

EXERCISES VIII. DIFFERENTIATION OF RATIONAL FUNCTIONS

Calculate the derivative of each of the following:

1. (*a*) $y=\frac{x-2}{x+4}\cdot$

(*b*) $y=\frac{x+1}{x-1}\cdot$

(*c*) $y=\frac{3x-4}{x^2+4}\cdot$

(*d*) $y=\frac{x}{1+x}\cdot$

(*e*) $y=\frac{1+x}{x}\cdot$

(*f*) $y=\frac{x^2+1}{x^2-1}\cdot$

(*g*) $y=\frac{x^2+x-3}{x^2-2x+6}\cdot$

(*h*) $y=\left(x^2+\frac{1}{x}\right)\div\left(1-\frac{1}{x}\right)\cdot$

2. (*a*) $y=\frac{3}{x^3}\cdot$

(*b*) $y=\frac{1}{5x^5}\cdot$

(*c*) $y=x^{-2}\left(=\frac{1}{x^2}\right)\cdot$

(*d*) $y=3x^{-4}$.

(*e*) $y=8x^{-7}$.

(*f*) $y=\frac{2}{x^3}+\frac{3}{x^2+1}\cdot$

(*g*) $y=2x^{-3}-3x^{-2}$.

(*h*) $y=4x^{-6}+\frac{6}{x^4}\cdot$

(*i*) $y=8x^{-10}+\frac{7}{x^{12}}-\frac{10}{x^4}\cdot$

(*j*) $y=ax^{-m}+bx^{-n}$.

3. (*a*) $v=\frac{u^2-1}{u^3-1}\cdot$

(*b*) $v=\frac{3u^5+5}{u^2}\cdot$

(*c*) $v=\frac{u^2}{u^3-1}\cdot$

(*d*) $v=\frac{1}{u-\frac{1}{u}}\cdot$

(*e*) $v=\frac{u^2+u+1}{u^2-u+1}\cdot$

(*f*) $s=\frac{1}{t^2+2t-3}-\frac{1}{t^2+t+6}\cdot$

(*g*) $s=\frac{2t+1}{t^3-1}-\frac{5}{t^2+t+1}\cdot$

(*h*) $s=a+\frac{b}{t+\frac{c}{t}}\cdot$

4. (*a*) $s=a+\frac{b}{t}-\frac{c}{t^2}\cdot$

(*b*) $p=\frac{1}{r}+\frac{1}{1+r}\cdot$

(*c*) $s=ht^4-kt^{-4}$.

(*d*) $p=r^2-\frac{r^4+1}{r^2-1}\cdot$

(*e*) $s=t^{-5}(t^3-2t^5+1)$.

(*f*) $q=(s^{-3}+4)(s^{-3}-5)$.

(*g*) $q=\frac{as^{-4}}{bs^2+c}\cdot$

(*h*) $p=\frac{r^2-r^3}{r^3-r^2}\cdot$

5. (*a*) $y=\frac{3+6x-5x^2}{x}\cdot$

(*b*) $y=\frac{16x^3+4x^2-3x}{3x^2/2}\cdot$

(*c*) $p=\frac{\theta^4/4+2\theta^3-6}{2\theta^2/7}\cdot$

(*d*) $p=\frac{z^3-2z+z^5-z^2}{z^2+z-2}\cdot$

6. Draw the following curves; obtain the equation of the tangent at the point indicated, and also at any point (x_0, y_0); determine the horizontal tangents if any exist, and show whether these points correspond to maxima or minima or neither.

(a) $y = \dfrac{1}{1-x^2}$; $x = -\dfrac{1}{2}$.

(b) $y = \dfrac{1-x}{1+x}$; $x = 2$.

(c) $y = \dfrac{1+x^2}{1-x^2}$; $x = \dfrac{1}{2}$.

(d) $y = \dfrac{1+x^2}{x}$; $x = -2$.

7. Compare the slopes of the curves $y = x$, $y = x^{-1}$ at the points at which they intersect. What is the angle between them?

8. Compare the slopes of the **family** of curves $y = x^n$, where $n = 0, +1, +2$, etc., $-1, -2$, etc., at the common point $(1, 1)$. What is the angle between $y = x^2$ and $y = x^{-1}$? See *Tables*, III, *A*.

21. Derivative of a Product. — The following rule is often useful in simplifying differentiations:

The derivative of the product of two functions is equal to the first factor times the derivative of the second plus the second factor times the derivative of the first:

$$\text{[VI]} \qquad \frac{d(u \cdot v)}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}.$$

If $y = u \cdot v$ where u and v are functions of x, a change Δx in x produces changes Δy, Δu, Δv in y, u, and v, respectively:

$$(A) \qquad y + \Delta y = (u + \Delta u)(v + \Delta v);$$

$$(B) \qquad \Delta y = (u + \Delta u)(v + \Delta v) - u \cdot v = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v;$$

$$(C) \qquad \Delta y/\Delta x = u(\Delta v/\Delta x) + v(\Delta u/\Delta x) + \Delta u \frac{\Delta v}{\Delta x};$$

$$(D) \qquad dy/dx = \lim_{\Delta x \doteq 0} (\Delta y/\Delta x) = u(dv/dx) + v(du/dx).$$

Example 1. To find the derivative of $y = (x^2 + 3)(x^3 + 4)$.

Method 1. We may perform the indicated multiplication and write:

$$\frac{dy}{dx} = \frac{d}{dx}[(x^2+3)(x^3+4)] = \frac{d}{dx}[x^5 + 3x^3 + 4x^2 + 12] = 5x^4 + 9x^2 + 8x.$$

Method 2. Using the new rule, we write:

$$\frac{dy}{dx} = (x^2+3)\frac{d}{dx}(x^3+4) + (x^3+4)\frac{d}{dx}(x^2+3)$$
$$= (x^2+3)3x^2 + (x^3+4)2x = 5x^4 + 9x^2 + 8x.$$

In other examples which we shall soon meet, the saving in labor due to the new rule is even greater than in this example.

22. The Derivative of a Function of a Function. Another convenient rule is the following:

The derivative of a function of a variable u, which itself is a function of another variable x, is found by multiplying the derivative of the original function with respect to u by the derivative of u with respect to x.

[VII] $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

If y is a function of u, and u is a function of x, a change Δx in x produces a change Δu in u; that in turn produces a change Δy in y; hence:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}.$$

Taking limits on both sides, we find:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$

This is really the same as the rule used in § 8, p. 14; for, if we divide both sides by du/dx, we find

[VII *a*] $$\frac{dy}{du} = \frac{dy}{dx} \div \frac{du}{dx},$$

which is the same as the rule of § 8, except that different letters are used.

Example 1. To find the derivative of $y = (x^2+2)^3$.

Method 1. We may expand the cube and write:

$$\frac{dy}{dx} = \frac{d}{dx}[(x^2+2)^3] = \frac{d}{dx}(x^6 + 6x^4 + 12x^2 + 8) = 6x^5 + 24x^3 + 24x.$$

Method 2. Using the new rule, we may simplify this work: let $u = x^2 + 2$, then $y = (x^2 + 2)^3 = u^3$; rule [VI] gives

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d(u^3)}{du} \cdot \frac{d(x^2+2)}{dx} = 3\,u^2\,(2\,x)$$

$$= 3(x^2+2)^2 \cdot (2\,x) = 3(x^4 + 4\,x^2 + 4) \cdot (2\,x) = 6\,x^5 + 24\,x^3 + 24\,x.$$

Example 2. If $y = t^2 + 2$ and $x = 3\,t + 4$, to find dy/dx.

Method 1. We may solve the equation $x = 3\,t + 4$ for t and substitute this value of t in the first equation:

$$y = \left(\frac{x-4}{3}\right)^2 + 2 = \frac{x^2}{9} - \frac{8}{9}x + \frac{34}{9}$$

$$\frac{dy}{dx} = \frac{2}{9}x - \frac{8}{9} = \frac{2}{9}(3\,t+4) - \frac{8}{9} = \frac{2\,t}{3}.$$

Method 2. Using the new rule (with letters as used in § 8, p. 14) we write:

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{d(t^2+2)}{dt} \div \frac{d(3\,t+4)}{dt} = 2\,t \div 3 = \frac{2}{3}t.$$

EXERCISES IX. — SHORT METHODS. RATIONAL FUNCTIONS

Calculate the derivative of each of the following:

1. (*a*) $y = 3\,x(x^2 + 1)$. (*d*) $y = (2\,x + 1)(1 - x + x^2)$.
(*b*) $y = x^3(x^2 + 3)$. (*e*) $y = (x^2 - 4)(1 + x^3)$.
(*c*) $y = (3\,x + 2)(2\,x - 3)$. (*f*) $y = (x^3 + 3\,x - 2)(x^2 - 2\,x)$.

2 (*a*) $y = (x^2 + 1)^2$. (*c*) $y = (1 - x^2)^2$. (*e*) $y = (a + bx)^2$.
(*b*) $y = (x^2 - 1)^3$. (*d*) $y = (1 - x^2)^3$. (*f*) $y = (a + bx)^3$.

3. (*a*) $y = (1 + 2\,x - 3\,x^2)^2$. (*d*) $y = (a + bx + cx^2)^3$.
(*b*) $y = (x^2 + 3\,x + 7)^3$. (*e*) $s = (3\,t^2 + 2\,t - 4)^4$.
(*c*) $s = (t^3 - t - 4)^2$. (*f*) $y = (a + bx + cx^2)^5$.

4. (*a*) $y = \dfrac{1}{(1 + 2\,x - 3\,x^2)^2}$. (*d*) $y = (2 + 3\,x^2)^{-2}\left[= \dfrac{1}{(2+3\,x^2)^2}\right]$.
(*b*) $s = \dfrac{1}{(1 - t^3)^2}$. (*e*) $v = (a + bx)^{-8}$.
(*c*) $y = \dfrac{x^3 + 3}{(x^2 + 2)^3}$. (*f*) $p = \dfrac{1}{(a - bs - cs^2)^3}$.

5. (*a*) $y = (1 - 5\,x^2)(3 - 4\,x^3)(1 - x)$. (*c*) $y = (x^2 + 2)^3(3\,x - 5)^2$.
(*b*) $y = x(x^2 + 3)(x^3 + 4)$. (*d*) $s = (t^3 - 2)^2(2\,t - 1)^3$.

6. Determine dy/dx in each of the following pairs of equations :

(a) $\begin{cases} y = 4u - 6u^2, \\ u = 3x - 4. \end{cases}$ (b) $\begin{cases} y = 6u^2 - 7u - 1, \\ u = x^2 - 1/2. \end{cases}$

(c) $\begin{cases} y = \dfrac{6z - 4z^3}{5z^2}, \\ z = 2 - 4x. \end{cases}$ (d) $\begin{cases} y = \dfrac{2z - 4}{z^2 - 1}, \\ z = \dfrac{9 - 4x^3}{2 - 4x}. \end{cases}$

7. Draw each of the curves represented by the following pairs of parameter equations and determine dy/dx:

(a) $\begin{cases} x = t^2, \\ y = 3t + 2. \end{cases}$ (b) $\begin{cases} x = 2t + 3t^2, \\ y = 2t + 4. \end{cases}$

What is the slope in each case when $t = 1$? Show this in your graphs. Find the value of the slope in each case at a point where the parameter has the value 2.

8. Draw the graph of the function $y = (2x - 1)^2(3x + 4)^2$. Determine its horizontal tangents.

9. Proceed as in Ex. 8, for the function $y = (2x - 1)^2 \div (3x + 4)^2$.

10. Show that if $y = (x - 1)^2(2x + 3)^2$, the derivative dy/dx has a factor $(x - 1)$ and a factor $(2x + 3)$; hence show that the given equation represents a curve tangent to the x-axis at $x = 1$ and at $x = -3/2$.

11. Show that if $y = (x - 2)^3(x^3 + 4x - 7)$, the derivative dy/dx has a factor $(x - 2)^2$. Show that the given curve is tangent to the x-axis at $x = 2$, but has no minimum or maximum there.

12. Apply the same reasoning which was used in Ex. 11 to the equation $y = (x - a)^n(x - b)^3$.

13. Show that the curve $y = x^3 + ax^2 + bx + c$ is tangent to the x-axis at $x = k$ if $(x - k)^2$ is a factor of the right-hand side.

14. Show that $y = P(x)$ where $P(x)$ is any polynomial, is tangent to the x-axis at $x = k$ if $(x - k)^2$ is a factor of $P(x)$.

23. Differentiation of Irrational Functions. In order to differentiate irrational expressions, we proceed to prove that the formula for the derivative of a power (Rule [IV]) holds true for all fractional powers:

[IV*a*] $$\frac{dx^n}{dx} = nx^{n-1}, \quad n = \pm\frac{p}{q}.$$

First proof. Let $y = x^{p/q}$, where p and q are positive integers; then, raising both sides to the power q,

(1) $$y^q = x^p.$$

If (x, y) and $(x + \Delta x,\ y + \Delta y)$ are pairs of related values of x and y, each pair must satisfy equation (1); hence (1) holds for x and y, and also

(2) $$(y + \Delta y)^q = (x + \Delta x)^p,$$

or

(3) $$y^q + q \cdot y^{q-1}\Delta y + (\text{several terms})\,\overline{\Delta y}^2 = x^p + px^{p-1}\Delta x + (\text{several terms})\,\overline{\Delta x}^2.$$

Subtracting (1) from (3), and dividing both sides of the resulting equation by Δx:

$$[qy^{q-1} + (\text{several terms})\,\Delta y]\,\frac{\Delta y}{\Delta x} = px^{p-1} + (\text{several terms})\,\Delta x,$$

or, $$\frac{\Delta y}{\Delta x} = \frac{px^{p-1} + (\text{several terms})\,\Delta x}{qy^{q-1} + (\text{several terms})\,\Delta y},$$

whence

$$\frac{dy}{dx} = \lim_{\Delta x \doteq 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{px^{p-1}}{qy^{q-1}} = \frac{px^{p-1}}{q\,(x^{p/q})^{q-1}} = \frac{p}{q}\,x^{p/q-1}.$$

This is the same as formula [IV] with $n = \dfrac{p}{q}$; hence [IV] holds for any positive fractional exponent.

That [IV] also holds for *negative* fractional exponents is now proved by means of Ex. 3, p. 33; hence [IV] holds for any positive or negative fractional exponent.

Second proof. Another proof will seem simpler to some students: if we set

(1) $$x = t^q, \text{ then } y = t^p,$$

which together are equivalent to $y = x^{p/q}$, and apply formula [VII$_a$] with suitable changes of letters, we find:

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = pt^{p-1} \div qt^{q-1} = \frac{p}{q}\,t^{p-q};$$

but since $t = x^{1/q}$, substitution for t gives

$$\frac{dy}{dx} = \frac{p}{q}(x^{1/q})^{p-q} = \frac{p}{q}x^{p/q-1}.$$

This proves [IV] for positive fractional values of n; the proof for negative fractional exponents is as given in Ex. 3, p. 33.

The rule also holds when n is incommensurable; for example, given $y = x^{\sqrt{2}}$, it is true that $dy/dx = \sqrt{2}\,x^{\sqrt{2}-1}$; we shall postpone the proof of this until § 84, p. 147.

24. Collection of Formulas. Any formula may be combined with [VII], for in any example, any convenient part may be denoted by a new letter, as in § 22. For example, Rule [IV] may be written

$$\frac{du^n}{dx} = \frac{du^n}{du}\cdot\frac{du}{dx}, \text{ by [VII]}, \qquad = nu^{n-1}\cdot\frac{du}{dx}, \text{ by [IV]}.$$

The formulas we have proved are collected here for easy reference:

[I] $$\frac{dc}{dx} = 0.$$

[II] $$\frac{d(c\cdot u)}{dx} = c\cdot\frac{du}{dx}.$$

[III] $$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$ Holds for subtraction also.

[IV] $$\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}.$$

[V] $$\frac{d\left(\frac{N}{D}\right)}{dx} = \frac{D\frac{dN}{dx} - N\frac{dD}{dx}}{D^2}.$$ Special case $$\frac{d\frac{c}{u}}{dx} = \frac{-c\frac{du}{dx}}{u^2}.$$

[VI] $$\frac{d(u\cdot v)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

[VII] $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}. \qquad [y = f(u),\ u = \phi(x)].$$

[VII$_a$] $$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}. \qquad [y = f(t),\ x = \phi(t)].$$

Special case: $\dfrac{dx}{dx} = 1. \quad (y = x.)$ (§ 19)

These formulas enable us to differentiate any simple algebraic function.

25. Illustrative Examples of Irrational Functions. In this article the preceding formulas are applied to examples.

Example 1. $\dfrac{d\sqrt{x}}{dx}=\dfrac{dx^{1/2}}{dx}=\dfrac{1}{2}x^{1/2-1}=\dfrac{1}{2}x^{-1/2}=\dfrac{1}{2\sqrt{x}}$. (See Ex. 4, p. 25.)

Example 2. Given $y=\sqrt{3x^2+4}$, to find dy/dx.

Method 1. Set $u=3x^2+4$, then $y=\sqrt{u}$; by Rule [VII],

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{2\sqrt{u}}\cdot 6x=\frac{6x}{2\sqrt{3x^2+4}}=\frac{3x\sqrt{3x^2+4}}{3x^2+4}.$$

Method 2. Square both sides, and take the derivative of each side of the resulting equation with respect to x:

$$\frac{d(y^2)}{dx}=\frac{d(3x^2+4)}{dx}=6x.$$

But by Rule [IV],

$$\frac{d(y^2)}{dx}=\frac{d(y^2)}{dy}\cdot\frac{dy}{dx}=2y\frac{dy}{dx};$$

hence,

$$2y\frac{dy}{dx}=6x,\quad\text{or}\quad\frac{dy}{dx}=\frac{3x}{y}=\frac{3x}{\sqrt{3x^2+4}}=\frac{3x\sqrt{3x^2+4}}{3x^2+4}.$$

This method, which is excellent when it can be applied, can be used to give a third proof of the Rule [IV] for fractional powers. The next example is one in which this method cannot be applied directly.

Example 3. Given $y=x^3-2\sqrt{3x^2+4}$, to find dy/dx.

$$\frac{dy}{dx}=\frac{d(x^3)}{dx}-2\frac{d}{dx}\sqrt{3x^2+4}=3x^2-\frac{6x\sqrt{3x^2+4}}{3x^2+4}.$$

Example 4. Given $y=(x^3-2)\sqrt{3x^2+4}$, to find dy/dx.

$$\frac{dy}{dx}=\sqrt{3x^2+4}\,\frac{d}{dx}(x^3-2)+(x^3-2)\frac{d}{dx}(\sqrt{3x^2+4})\qquad\text{[by Rule VI]}$$

$$=\sqrt{3x^2+4}\cdot 3x^2+(x^3-2)\frac{3x\sqrt{3x^2+4}}{3x^2+4}\qquad\text{[by Example 2]}$$

$$=\sqrt{3x^2+4}\left[3x^2+(x^3-2)\cdot\frac{3x}{3x^2+4}\right]=\sqrt{3x^2+4}\cdot\frac{12x^4+12x^2-6x}{3x^2+4}.$$

Example 5. Given $y=\dfrac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}$, to find dy/dx.

First reduce y to its simplest form :

$$y=\frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\cdot\frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}=\frac{2\,x+1-2\sqrt{x^2+x}}{(x+1)-x}=2\,x+1-2\sqrt{x^2+x}.$$

Then

$$\frac{dy}{dx}=\frac{d}{dx}(2\,x+1)-2\,\frac{d}{dx}\sqrt{x^2+x}=2-2\,\frac{d\sqrt{u}}{du}\,\frac{du}{dx},$$

where $u=x^2+x$; hence

$$\frac{dy}{dx}=2-2\,\frac{1}{2\sqrt{u}}\,\frac{du}{dx}=2-\frac{1}{\sqrt{x^2+x}}\,(2\,x+1).$$

This example may be done also by first applying the rule for the derivative of a fraction [Rule V] ; but the work is usually simpler, as in this example, if the given expression is first simplified.

EXERCISES X. — ALGEBRAIC FUNCTIONS

Calculate the derivatives of

1. (a) $y=x^{4/3}$. (d) $y=\sqrt{x^3}$. (g) $y=\sqrt{x\sqrt{x}}$.

(b) $s=10\,t^{5/2}$. (e) $s=2\sqrt[3]{x^2}$. (h) $y=6\,x^{-2/3}$.

(c) $y=x^{1/5}$. (f) $v=\frac{4}{5}\sqrt[5]{u^4}$. (i) $s=7\,t^{-3/4}$.

2. (a) $y=x^2\dfrac{5}{\sqrt{x^6}}-\dfrac{10\,x^3}{\sqrt{x^5}}$. (b) $v=\dfrac{6}{u^4}-\dfrac{2}{\sqrt[3]{u}}$.

(c) $s=t^3(2\,t^{2/3}+3\,t^{-2/3})$. (d) $y=2\sqrt[3]{x}(x^{1/3}+x^{5/3})$.

(e) $y=\dfrac{2}{x^3}-\dfrac{x\sqrt{x^3}}{5}+\sqrt[3]{x^2}$. (f) $s=t^3\left(\dfrac{2}{11}t^2\sqrt{t}-\dfrac{72}{23}\dfrac{t}{\sqrt[6]{t}}+\dfrac{16}{3}\right)$.

3. $y=\frac{24}{11}\,x\sqrt[6]{x^5}-\frac{18}{17}\,x^2\sqrt[6]{x^5}+\frac{3}{10}\,x^3\sqrt[3]{x}$.

4. (a) $y=\sqrt{2+3\,x}$. (g) $y=\sqrt[3]{1+x^2}$.

(b) $s=\sqrt{3\,t-4}$. (h) $y=\sqrt[4]{2\,x^2+4\,x}$.

(c) $v=u\sqrt{2+3\,u}$. (i) $y=x^2\sqrt{3\,x-4}$.

(d) $s=\sqrt{t^2-1}$. (j) $y=(5+3\,x)\sqrt{6\,x-4}$.

(e) $s=\sqrt{t^2-3\,t}$. (k) $v=\sqrt{1-x+x^2}$.

(f) $s=\dfrac{5+3\,t}{\sqrt{6\,t-5}}$. (l) $s=\dfrac{\sqrt{6\,t-5}}{5+3\,t}$.

5. (a) $y=\sqrt{1+\sqrt{x}}$. (b) $s=\sqrt{\dfrac{1-t^2}{1+t^2}}$. (c) $v=\dfrac{\sqrt{a^2-u}}{u}$.

6. (a) $y = (9 - 6x + 5x^2)\sqrt[3]{(1 + x^4)^2}$. (b) $s = (1 + t^2)\sqrt{1 - t^2}$.

7. (a) $y = \dfrac{2}{3x^3}\sqrt{7x^2 - 9}$. (b) $v = \left(\dfrac{3}{u^7} + \dfrac{2}{u^5}\right)\sqrt{(3 - 5u^2)^5}$.

8. (a) $y = \dfrac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$. (First rationalize the denominator.)

(b) $y = \dfrac{\sqrt{1+x^2} + x}{\sqrt{1+x^2} - x}$. (c) $y = \dfrac{\sqrt{a^2 - x^2}}{\sqrt{a^2 + x^2}}$.

9. (a) $y = \dfrac{\dfrac{9}{320}x^4 - \dfrac{1}{8x^2}}{\sqrt[3]{(20 - 3x^6)^2}}$. (b) $y = (\sqrt{x^3} - \frac{1}{40})\sqrt[3]{(\frac{1}{4} + 6\sqrt{x^3})^5}$.

10. Draw the graphs of the equations below, and determine the tangent at the point mentioned in each case.

(a) $y = \sqrt{1 - x^2}$, $(x = \frac{3}{5})$. (d) $y = \sqrt{(1+x)(2+3x)}$, $(x = 2)$.
(b) $y = \sqrt{1 + x^2}$, $(x = \frac{4}{3})$. (e) $y = x\sqrt{1 + x}$, $(x = 1)$.
(c) $y = \sqrt{x}$, $(x = 2)$. (f) $y = x^{1/2} - x^{1/3}$, $(x = 1)$.

11. Find the angle between the curves $y = x^{1/2}$ and $y = x^2$ at each of their points of intersection.

12. Find the angle between the curves $y = x^{2/3}$ and $y = x^{3/2}$ at $(1, 1)$.

13. Find the angle between the curves $y = x^{p/q}$ and $y = x^{q/p}$ at $(1, 1)$.

14. In compressing air, if no heat escapes, the pressure and volume of the gas are connected by the relation $pv^{1.41} = \text{const.}$ Find the rate of change of the pressure with respect to the volume, dp/dv.

15. In compressing air, if the temperature of the air is constant, the pressure and the volume are connected by the relation $pv = \text{const.}$ Find dp/dv, and compare this result with that of Ex. 14.

16. Find dy/dx for $y = x^2$; for $y = x^{2.3}$; for $y = x^{2.5}$; for $y = x^{2.8}$; for $y = x^3$. Show that the value of dy/dx increases steadily in each case as x increases, and that the magnitudes of the derivatives are in the order of the exponents at $x = 1$ and for all larger values of x.

17. Draw a graph to show the values of the *derivatives* for each of the curves of Ex. 16; find graphically the values of x for which the derivative of each of them is the same as that of $y = x^2$.

PART II. EQUATIONS NOT IN EXPLICIT FORM DIFFERENTIALS

26. Solution of Equations. An equation in two variables x and y is often given in unsolved form; *i.e.* neither variable is expressed directly in terms of the other. Thus the equation

(1) $$x^2+y^2=1$$

represents a definite relation between x and y; graphically, it represents a circle of unit radius about the origin.

Such an equation often can be solved for one variable in terms of the other; thus (1) gives

(2) $$y=\sqrt{1-x^2}, \text{ or } y=-\sqrt{1-x^2}.$$

The first solution represents the upper half of the circle, the second the lower half. From this solution, we can find dy/dx as in § 25:

(3) $$\frac{dy}{dx}=\frac{-x}{\sqrt{1-x^2}}, \text{ or } \frac{dy}{dx}=\frac{+x}{\sqrt{1-x^2}},$$

where the first holds true on the upper half, the second on the lower half, of the circle.

By Rule [VII] such a derivative may be found directly **without solving** the equation. From (1)

$$\frac{d}{dx}(x^2+y^2)=\frac{d1}{dx}=0;$$

but $$\frac{d}{dx}(x^2+y^2)=\frac{d(x^2)}{dx}+\frac{d(y^2)}{dx}=2x+\frac{d(y^2)}{dy}\cdot\frac{dy}{dx}, \quad \text{by VII};$$

hence

(4) $$2x+2y\frac{dy}{dx}=0,$$

or

(5) $$\frac{dy}{dx}=-\frac{x}{y}.$$

This result agrees with (3), since $y=\pm\sqrt{1-x^2}$.

This method is the same as that used in the first proof of [IV_a] in § 23, p. 39, and also in the second solution of Ex. 2,

p. 41. It may be used whenever the given equation really has any solution, without actually getting that solution.

Such a formula as (4) is much more convenient than (3), since it is more compact, and is stated in one formula instead of in two. But the student must never use (5) for values of x and y without substituting those values in (1) to make sure that the point (x, y) actually lies on the curve; and he must never use (5) when (5) does not give a definite value for dy/dx.* Thus it would be very unwise to use (4) at the point $x=1$, $y=2$, for that point does not lie on the curve (1); it would be equally unwise to try to substitute $x=1$, $y=0$, since that would lead to a division by zero, which is impossible.

27. Explicit and Implicit Functions. If one variable y is expressed directly in terms of another variable x, we say that y is an **explicit** function of x.

If, as in § 26, the two variables are related to each other by means of an equation which is not solved explicitly for y, then y is called an **implicit** function of x. Thus, (1) in § 26 gives y as an implicit function of x; but either part of (2) gives y as an explicit function of x.

If an equation in x and y is given, so that y is an implicit function of x, we may either solve that equation for y, as in the first part of § 26, and then differentiate as we have done up to this point; or we may proceed to find the derivative without solving, by means of Rule [VII], as in § 26. The latter method is especially fortunate when the given equation is difficult to solve.

Definition. If the original equation is a simple polynomial in x and y equated to zero, any explicit function of x obtained by solving it for y is called an **algebraic function**. See § 18.

* These precautions, which are quite easy to remember, are really sufficient to avoid all errors for all curves mentioned in this book, at least provided the equation like (4) [not (5)] is used in its original form, before any cancellation has been performed.

Example 1. $x^3+y^3-3xy=0$. (Folium of Descartes : *Tables*, III, I_5.)

This equation is difficult to solve directly for y. Hence, as in § 26, we find dy/dx by Rule [VII]; differentiating both sides with respect to x, we find :

$$3x^2+3y^2\frac{dy}{dx}-3y-3x\frac{dy}{dx}=0;$$

whence
$$\frac{dy}{dx}=\frac{y-x^2}{y^2-x}.$$

At the point (2/3, 4/3), for example, $dy/dx=4/5$; hence the equation of the tangent at (2/3, 4/3) is $(y-4/3)=(4/5)(x-2/3)$ or $4x-5y+4=0$. Verify the fact that the point (2/3, 4/3) really lies on the curve. Note that this formula is useless at the point (0, 0) although that point lies on the curve.

28. Inverse Functions. If y is given as an *explicit* function of x,

(1)
$$y=f(x),$$

and if this equation can be solved for x in terms of y,

(2)
$$x=\phi(y),$$

then $\phi(y)$ is called the **inverse** function of $f(x)$. If this solution is substituted in the original equation (1), that equation must be satisfied:

(3)
$$y=f[\phi(y)].$$

Thus, if $y=x^3$; we find $x=y^{1/3}$; substituting $y^{1/3}$ for x in the original equation gives $y=(y^{1/3})^3$, which is an identity.

Since
$$\frac{\Delta y}{\Delta x}=1\div\frac{\Delta x}{\Delta y},$$

it follows that

[VII *b*]
$$\frac{dy}{dx}=1\div\frac{dx}{dy},$$

unless $dx/dy=0$.* This rule is really a special case of Rule [VII]; for if, in Rule [VII], $y=x$, we get

* The precautions to be observed are exactly the same as those of § 26.

$$\frac{dx}{du} \cdot \frac{du}{dx} = 1,$$

which agrees with [VII $_b$] except that different letters are used.

Thus if $u = x^3$, $x = u^{1/3}$; $du/dx = 3\,x^2$, $dx/du = 1/(3\,u^{2/3})$; then $(du/dx) \cdot (dx/du) = (3\,x^2) \cdot [1/(3\,u^{2/3})] = 1$ since $x = u^{1/3}$.

29. Parameter Forms. If both x and y are given as explicit functions of a third variable t:

(1) $$x = f(t),\ y = \phi(t),$$

we call t a **parameter**, and the equations (1) **parameter equations**. If we can eliminate t, we obtain an equation connecting x and y directly:

(2) $$F(x, y) = 0.$$

From (2) we might find dy/dx as in § 26; but it is usually easier to proceed as in § 8, p. 14, and § 22, p. 36, using the formula [VII $_a$], in the letters x, y, t:

[VII $_a$] $$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}.$$

Thus in Example 2, p. 37, we found dy/dx by this formula from equations like (1); first, by eliminating t; second, by using [VII $_a$].

EXERCISES XI.—FUNCTIONS NOT IN EXPLICIT FORM

In each of these exercises the student should take some point on the curve, and find the equation of the tangent there.

1. From the equation $xy = 1$ find dy/dx by the two methods of § 26, first solving for y, then without solving for y. Write the result in terms of x and y; and also in terms of x alone, when possible.

2. Find dy/dx in the following examples by the two methods of § 26:

(a) $x^2y = 10$.

(b) $x^2 + xy - 5 = 0$.

(c) $x^2 - y^2 = 1$.

(d) $xy + x + y = 0$.

(e) $4\,x^2 - y^2 = 16$.

(f) $x^2 - 2\,xy + 2\,x - 3\,y + 4 = 0$.

(g) $x^3 - x^2y - 4 = 0$.

(h) $x^3 - y^2 = 0$.

(i) $x^3 + y^3 = a^3$.

(j) $x^3 - y^3 = a^3$.

3. Find dy/dx in the following examples without solving for y; check the answers when possible by the other method of § 26:

(a) $x^2 + 3xy + y^2 = 2.$

(b) $x^2y^2 + 2xy + 7 = 0.$

(c) $ax^2 + 2hxy + by^2 = k.$

(d) $y^4 - 2y^2x + x^2 = 0.$

(e) $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$

(f) $\sqrt{x} + \sqrt{y} = \sqrt{a}.$

(g) $x^{3/2} + y^{3/2} = a^{3/2}.$

4. Find the inverses of the following functions by solving the equations for x; then find dx/dy. Verify that $(dx/dy)(dy/dx) = 1$ in each case.

(a) $y = 2x + 3.$

(b) $y = 5 - x.$

(c) $y = \dfrac{3}{4x}.$

(d) $y = 6 - \dfrac{2}{x}.$

(e) $y = \dfrac{2 - x}{2x + 3}.$

(f) $y = \sqrt{1 - x^2}.$

(g) $y = \sqrt{a^2 + x^2}.$

(h) $y = \dfrac{a}{\sqrt{1 + x}}.$

(i) $y = \dfrac{ax + b}{cx + d}.$

(j) $y = x^3.$

(k) $y = \frac{1}{2}x^2 + 3.$

(l) $y = 3x^2 - 5.$

(m) $y = x^3 + 2.$

(n) $y = \dfrac{x^2 + 2}{x^2 - 2}.$

(o) $y = (x - 1)(x + 2).$

5. In the following examples find dy/dx without solving for y:

(a) $x = \dfrac{2y + 2}{3y - 1}.$

(b) $x = y^2 - 2y + 4.$

(c) $x = y^3 + 5.$

(d) $x = y^3 - 3y + 7.$

(e) $x = \dfrac{y^2 + 2y - 3}{y^2 + 2}.$

(f) $x = y\sqrt{1 + y^2}.$

(g) $x = y^2 + \sqrt{1 - y^2}.$

(h) $x = \dfrac{y + \sqrt{1 + y}}{y - \sqrt{1 - y}}.$

(i) $x = \dfrac{y^3 + 1}{y^3 - 1}.$

6. In the following pairs of parameter equations, find dy/dx by § 29; when possible eliminate t to find the ordinary equation, and show that the derivative found is correct.

(a) $\begin{cases} x = 4t, \\ y = 16t^2. \end{cases}$

(b) $\begin{cases} x = 2t - 5, \\ y = 3t + 2. \end{cases}$

(c) $\begin{cases} x = 3t^2 - 1, \\ y = 2t^3. \end{cases}$

(d) $\begin{cases} x = \dfrac{3t}{1 + t^3}, \\ y = \dfrac{3t^2}{1 + t^3}. \end{cases}$

(e) $\begin{cases} x = \dfrac{t^2 - 1}{t^2 + 1}, \\ y = \dfrac{-2t}{t^2 + 1}. \end{cases}$

(f) $\begin{cases} x = \dfrac{t^2 + 1}{t^2 - 1}, \\ y = \dfrac{2t}{t^2 - 1}. \end{cases}$

7. In each of the problems of Ex. 6, find the horizontal speed and the vertical speed of a body which moves as stated there, x and y representing the coördinates of the body at the time t. The total **speed along the curve** is the square root of the sum of the squares of these two; find this total speed in each case.

8. In a circle of unit radius about the origin $dy/dx = -x/y$; this is positive when x and y have different signs, negative when x and y have the same sign. Show that this agrees with the fact that the circle rises in the second and fourth quadrants and falls in the first and third quadrants as x increases.

9. Show that the curve $xy = 1$ is falling at all its points.

10. Show that the curve $x^2y = 1$ is rising in the second quadrant and falling in the first quadrant.

11. The equation $x^{1/2} + y^{1/2} = 1$ is the equivalent to the equation $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$, if the radicals $x^{1/2}$ and $y^{1/2}$ be taken with both signs. Show that the values of dy/dx calculated from the two equations agree. By methods of analytic geometry, it is easy to see that the curve is a parabola whose axis is the line $y = x$, with its vertex at $(1/4, 1/4)$.

12. The curve of Ex. 11 is also represented by the parameter equations $4x = (1 + t)^2$, $4y = (1 - t)^2$. Test this fact by substitution, and show that the value of dy/dx obtained from these equations agrees with the value obtained in Ex. 11. [The curve is most easily drawn from the parameter equations.]

If t denotes the time in seconds since a particle moving on this curve passed the point $(1/4, 1/4)$, find the total speed of the particle at any time. (See Ex. 7.)

30. Rates. In using the notation dy/dx for a derivative, we called attention to the fact that this symbol does not represent a fraction, but rather the limit of a fraction; $dy/dx = \lim \Delta y/\Delta x$.

We may, however, think of any quantity as a fraction by simply providing it with a convenient denominator; thus $3 = 12 \div 4$, which is a very convenient way of writing 3 if we wish to add it to $1/4$.

In the case of any rate of change, it is very usual to do this;

thus a speed, even though it be thought of as instantaneous, is usually told in feet per second, *i.e.* it is mentioned as if it were an average speed over a whole second. A slope — even of a curve at a point — is spoken of as the tangent of an angle, which, by definition, is the ratio of one distance to another distance. The death rate in a city or in a state is usually given per 100,000 inhabitants, though it is understood that the city does not have exactly 100,000 inhabitants. Even the death rate due to a particular disease — say appendicitis — is quoted *per* 100,000; the statement that 98.4 persons per 100,000 die annually, does not mean that 98.4 in any given 100,000 die, for the number of deaths is clearly an integer; the denominator 100,000 is used solely for convenience and for the purpose of ready comparison between one city and another, or between one disease and another.

Rates are usually stated in some such convenient manner. As in the case of death rate, such a common denominator is useful in all comparisons between different rates of change of the same character; to compare a speed of 56 feet per second with a speed of 40 miles per hour it is highly desirable to reduce them to a common denominator, and to express both of them, for example, in feet per second.

31. The Differential Notation. A device of exactly this character is often convenient in our symbol for a derivative; if we are dealing, for example, with the slope of a curve, we have

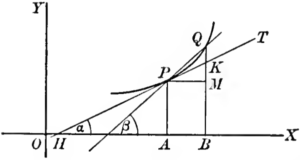


Fig. 12.

$$(1)\ m = \tan \alpha = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

where α is the angle XHT. In this case a convenient denominator is already in the figure; for in the triangle MPK,

$$(2) \qquad m = \tan\alpha = \tan XHT = \tan MPK = \frac{MK}{PM} = \frac{MK}{\Delta x},$$

where $\Delta x = PM = AB$.

This results in throwing m into the form of a fraction, with a denominator Δx, a quantity with which we are quite familiar; Δx means, as before, the difference of any two values of x, and this may be any amount we desire except zero.

The new quantity MK, the height of the triangle MPK, is called the **differential** of y, and it is denoted by the symbol $\boldsymbol{dy}$; its value is

$$(3) \qquad \boldsymbol{dy = m\,\Delta x},$$

which varies for different values of Δx.

In particular, if the curve is the straight line $y = x$, we find $m = 1$; hence the differential of x is

$$(4) \qquad \boldsymbol{dx = 1 \cdot \Delta x}.$$

If we divide (3) by (4) we find

$$(5) \qquad dy \div dx = m,$$

where $dy \div dx$ now denotes a real division, since dy and dx are actual quantities defined by the equations (3) and (4), and $dx\ (= \Delta x)$ is not zero.

Since m stands for the derivative of y with respect to x, it follows that that derivative is equal to the quotient of dy by dx,

$$(6) \qquad \boldsymbol{\frac{dy}{dx} = dy \div dx};$$

this fact is the reason for our use of the symbol dy/dx to represent a derivative originally.

In the figure all quantities here mentioned are shown:

$$dx = \Delta x = AB,\ dy = MK,\ \Delta y = MQ,\ \frac{\Delta y}{\Delta x} = \tan\beta,\ \frac{dy}{dx} = \tan\alpha.$$

$MK = dy$ is the change that would have taken place in y, for the change $AB = dx$ in x, if dy/dx, the instantaneous rate of change (or the slope at P), had been maintained. The quan-

tities $dx(=\Delta x)$,* $dy(=m\,\Delta x)$, Δy, $\Delta y - dy\,(=KQ)$, are infinitesimal when Δx approaches zero, *i.e.* they approach zero as Δx approaches zero.

32. Differential Formulas. For any given function $y=f(x)$, dy can be computed in terms of $dx\,(=\Delta x)$, by computing the derivative and multiplying it by dx. Thus, if $y=x^2$, $m=dy/dx=2\,x$, and $dy=m\,dx=2\,x\,dx$; again, if $y=x^3-12\,x+7$, $m=3\,x^2-12$ and $dy=m\,dx=(3\,x^2-12)\,dx$.

Every formula for differentiation can therefore be written as a **differential formula**; the first six in the list in § 24, p. 40, become after multiplication by dx:

[I] $dc=0$. (The differential of a constant is zero.)

[II] $d\,(c\cdot u)=c\cdot du.$

[III] $d(u+v)=du+dv.$

[IV] $d(u^n)=nu^{n-1}du.$

[V] $d\left(\dfrac{N}{D}\right)=\dfrac{DdN-NdD}{D^2}.$

[VI] $d(u\cdot v)=u\,dv+v\,du.$

Rules [VII], [VII_a], of § 24, p. 40, and [VII_b], of § 28, p. 46, appear as identities, since the derivatives may actually be used as quotients of the differentials. From the point of view of the differential notation Rule [VII] merely shows that we may use algebraic cancellation in products or quotients which contain differentials.

Rules [I]–[VI] are sufficient to express all differentials of simple algebraic functions. A great advantage occurs in the case of equations not in explicit form, since all applications of Rule [VII] reduce to algebraic cancellation of differentials.

* This equation does not assign any particular value to dx but only makes it coincide with the value of Δx chosen above. While we usually think of an infinitesimal as small, because at last it always becomes small, any particular value of an infinitesimal is a fixed finite quantity and may be chosen at pleasure.

Example 1. Given $y = x^3 - 12x + 7$, to find dy and m.

$$dy = d(x^3 - 12x + 7) = d(x^3) - d(12x) + d(7) = 3x^2dx - 12\,dx,$$

whence $m = dy \div dx = 3x^2 - 12$ as in Example 3, p. 10.

Example 2. Given $y = \dfrac{x^2+3}{3x-7}$, to find dy (Example 1, p. 33).

$$dy = \frac{(3x-7)d(x^2+3) - (x^2+3)\,d(3x-7)}{(3x-7)^2}$$
$$= \frac{(3x-7)\cdot 2x - (x^2+3)\cdot 3}{(3x-7)^2}dx.$$

Example 3. Given $y = (x^2+2)^3$, to find dy (Example 1, p. 36).

$$dy = d[(x^2+2)^3] = 3(x^2+2)^2\,d(x^2+2)$$
$$= 3(x^2+2)^2 \cdot 2x \cdot dx.$$

Example 4. Given $y = x^3 - 2\sqrt{3x^2+4}$, to find dy (Example 3, p. 41).

$$dy = d(x^3) - 2\,d\sqrt{3x^2+4}$$
$$= 3x^2dx - 2\frac{1}{2\sqrt{3x^2+4}} \cdot d(3x^2+4)$$
$$= \left(3x^2 - \frac{1}{\sqrt{3x^2+4}}6x\right)dx.$$

Example 5. Given $x^2 + y^2 = 1$, to find dy in terms of dx (§ 26, p. 44).

$$d(x^2+y^2) = d(1) = 0\text{; but } d(x^2+y^2) = d(x^2) + d(y^2)$$
$$= 2\,xdx + 2\,ydy\text{;}$$

hence $2x\,dx + 2y\,dy = 0$, or $dy = -(x/y)\,dx$, or $m = dy/dx = -x/y$.

Example 6. To find dy and m when $x^3 + y^3 - 3xy = 0$ (Example 1, p. 46).

$$d(x^3) + d(y^3) - 3\,d(xy) = 0,$$

or

$$3x^2\,dx + 3y^2\,dy - 3x\,dy - 3y\,dx = 0,$$

or

$$(x^2 - y)\,dx + (y^2 - x)\,dy = 0,$$

whence

$$dy = \frac{y-x^2}{y^2-x}dx, \text{ or } m = \frac{dy}{dx} = \frac{y-x^2}{y^2-x}.$$

Example 7. To find dy in terms of dx when $x = 3t + 4$, $y = t^2 + 2$ (Example 2, p. 37).

We find $dx = d(3t+4) = 3\,dt$; $dy = d(t^2+2) = 2t\,dt$;

hence $m = dy \div dx = (2/3)\,t$, or $dy = (2/3)\,t\,dx$;

but since $t = (x-4)/3$, this may be written:

$$dy = (2/9)(x-4)dx, \text{ or } m = \frac{dy}{dx} = (2/9)(x-4).$$

EXERCISES XII. — DIFFERENTIALS

[These exercises may be used for further drill in differentiation, and for reviews. It is scarcely advisable that all of them should be solved on first reading.]

Calculate the differentials of the following expressions:

1. (a) $y = a + 2bx + cx^2$. (b) $y = (a + x^2)^3$.
(c) $y = (a - bx^3)^5$. (d) $y = (a + bx - cx^2)^2$.

2. (a) $y = \left(a + \dfrac{1}{x}\right)^3$. (b) $y = \dfrac{1}{a + bx}$.
(c) $y = \dfrac{1}{(a - bx)^2}$. (d) $y = \dfrac{1}{(a + bx + cx^2)^2}$.

3. (a) $y = x^2(a - x)^3$. (b) $y = (1 - 2x)(1 + 3x)$.
(c) $y = x^4(a - 2x^3)^2$. (d) $(a - bx)^2(c + dx^2)$.

4. (a) $y = \sqrt{2x + x^2}$. (b) $s = t\sqrt{1 - t^2}$.
(c) $y = \sqrt{1 - x^4}$. (d) $y = (a + x)\sqrt{a - x}$.

5. (a) $s = t\sqrt{1 + t}$. (b) $s = t^2\sqrt{a - t}$.
(c) $y = \dfrac{1}{\sqrt{2x + x^2}}$. (d) $s = \dfrac{1 - t}{1 + t}$.

6. (a) $q = \dfrac{r^2 + 2r + 1}{r^3 - 1}$. (b) $q = \dfrac{1 - 2r^2}{2 - r^2}$.
(c) $q = \dfrac{r^2 - 2r + 3}{r^2 + 2r - 3}$. (d) $y = \dfrac{z^3 - 5z^7 + 2z}{z^8 + 3 - z}$.

7. (a) $y = \dfrac{1}{\sqrt[3]{(2 - \theta^3)^4}}$. (b) $y = \dfrac{1}{\sqrt[4]{(\theta - a)^3}}$.
(c) $y = \dfrac{\sqrt{a + b\theta}}{\theta}$. (d) $y = \dfrac{\theta}{\sqrt{1 + \theta}}$.

8. (a) $y = (a + bx^n)^p$. (b) $y = \sqrt[p]{a + bx^n}$.
(c) $y = \dfrac{1}{(a + bx^n)^p}$. (d) $y = \dfrac{1}{\sqrt[p]{a + bx^n}}$.

9. (a) $z = \dfrac{1 + \sqrt{y}}{1 - \sqrt{y}}$. (b) $z = \sqrt{\dfrac{a + by}{a - by}}$.
(c) $z = \sqrt{\dfrac{y}{y^2 - a^2}}$. (d) $z = \dfrac{1}{(a + bx)^{3/2}}$.

10. (a) $v = (\frac{10}{3} + u^2)\sqrt{(5 + 2u)^3}$. (b) $v = \left(7 - \frac{6}{u^2}\right)\sqrt[7]{\left(3 + \frac{6}{u^2}\right)^3}$.

11. (a) $\theta = \frac{\alpha^3 - 1}{\alpha - 1}$. (b) $\theta = \frac{\alpha^3 + a\alpha^2 + a\alpha + 1}{\alpha + 1}$.

(c) $\theta = \alpha(a^2 + \alpha^2)\sqrt{a^2 - \alpha^2}$. (d) $\theta = \alpha^3(a^2 - \alpha^2)^{3/2}$.

12. (a) $y = (a + bx)^{-1}$. (b) $y = (a + bx)^{-2}$.
(c) $y = (a + bx^2)^{-1}$. (d) $y = (a + bx^2)^{-2}$.

13. (a) $z = \left(\frac{1 + y^{-1}}{1 - y^{-1}}\right)^2$. (b) $z = \left(\frac{1 + y^2}{1 - y^2}\right)^{-1}$.
(c) $z = (Ay^{-3} + By^{-5})^2$. (d) $z = (Ay^{-3} + By^{-5})^{-2}$.
(e) $y = A(a + bx)^{10} + B(a - bx)^{-10}$.

14. Determine dy in terms of dx from the equations below:

(a) $Ax + By + C = 0$. (g) $\sqrt{x} + \sqrt{y} = c$.
(b) $xy + y = 1$. (h) $(1 - ax)(x^2 + y^2) = 4$.
(c) $x^2 - 2xy - 3y^2 = 0$. (i) $x^2 + y^2 = (ax + b)^2$.
(d) $\frac{x^2 + y^2}{x^2 - y^2} = x^2$. (j) $\frac{y^2}{x^2} = \frac{a + bx}{a - bx}$.
(e) $y^2(x - a) = (x + a)^3$. (k) $y^5 - 5axy + x^5 = 0$.
(f) $y^4 - 2y^2x - 1 = 0$. (l) $(x + y)^{3/2} + (x - y)^{3/2} = a^{3/2}$.

15. Obtain the equation of the tangent at $(2, -1)$ to the curve

$$4x^2 - 2xy - 5y^2 - 6x - 4y - 7 = 0.$$

16. Obtain the equation of the tangent at $(2, 1)$ to the curve

$$x^3 - 7x^2y - 5y^3 + 4x^2 - 10xy + 8x - 5y + 18 = 0.$$

17. Obtain the equation of the tangent at (x_0, y_0) to each of the following curves:

CURVE	TANGENT
(a) $y^2 = 4ax$;	$yy_0 = 2a(x + x_0)$.
(b) $x^2 + y^2 = a^2$;	$xx_0 + yy_0 = a^2$.
(c) $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$;	$\frac{xx_0}{a^2} \pm \frac{yy_0}{b^2} = 1$.
(d) $(x + y)^2 = 1$;	$(x + y)(x_0 + y_0) = 1$.

18. Find the derivative dy/dx for the curves defined by each of the pairs of parameter equations given below:

(a) $\begin{cases} x = \dfrac{2t}{1+t}, \\ y = \dfrac{1-t}{1+t}. \end{cases}$ (b) $\begin{cases} x = \dfrac{2}{3}\sqrt{2t^3}, \\ y = \dfrac{1}{2}t^2. \end{cases}$ (c) $\begin{cases} x = \dfrac{3a-2t}{at}, \\ y = \dfrac{4(a-t)^3}{a^2t^2}. \end{cases}$

(d) $\begin{cases} x = \dfrac{\theta}{1+\theta}, \\ y = \theta^{-1} + \theta^{-2}. \end{cases}$ (e) $\begin{cases} x = 4\pi r^2, \\ y = \dfrac{4}{3}\pi r^3. \end{cases}$ (f) $\begin{cases} x = \dfrac{1}{4\pi r^2}, \\ y = \dfrac{3}{4\pi r^3}. \end{cases}$

19. If a particle moves so that its coördinates (x, y) at any time t are

$$x = \frac{2t}{1+t^2}, \quad y = \frac{1-t^2}{1+t^2},$$

show on the same diagram the values of x and of y in terms of time; what are the extreme values of x and of y, and when are they attained? From the diagram construct another showing the (x, y) curve followed by the particle.

20. Calculate the x and y components of the speed (v_x and v_y) at any time t, and the resultant speed $\sqrt{v_x^2 + v_y^2}$, along the path, in the motion of Ex. 19. Show that $v_y \div v_x = dy/dx$. See Ex. 7, p. 49.

21. If a particle moves so that its coördinates in terms of the time are

$$x = 1 - t + t^2, \quad y = 1 + t + t^2,$$

show that its path is a parabola. Show that from the moment $t = 0$ its speed steadily increases.

22. A point moves on a straight line so that its distance s from a fixed point on the line at any time t is as given below. Describe the motion from $t = 0$, giving the times when the speed is positive, negative, zero. Draw the (s, t) diagrams and the (v, t) diagrams.

(a) $s = t^2 - 4t + 3.$
(b) $s = t^3 - 15t^2 + 63t - 4.$
(c) $s = 3t^4 - 40t^3 + 54t^2 - 10.$

23. If the volume of a sphere increases at the rate of 2 cu. ft. a second, calculate the rate of change per second of the radius and of the surface. What are these rates when the volume is 100 cu. ft.?

24. If R denotes the radius of a sphere, S the surface, and V the volume, calculate the differential of each of these in terms of each of the others.

25. If the radius of a cylinder expands at the rate of 1/2 in. a second, starting with a value 5 in., and if the height remains fixed at 10 in., at what rate per second is the volume changing at any time t? When $t = 10$ sec.? The same for the total surface?

26. When you walk straight away from a street lamp with uniform speed, does the end of your shadow also move with uniform speed? Supposing that your height is 70 in., show how fast the shadow tip moves if you walk 6 ft. per second away from a lamp 10 ft. above ground.

27. The electrical resistance of a platinum wire varies with the temperature, according to the equation

$$R = R_0(1 - a\theta + b\theta^2)^{-1};$$

calculate dR in terms of $d\theta$. What is the meaning of $dR/d\theta$?

28. Van der Waal's equation giving the relation between the pressure and volume of a gas at constant temperature is

$$\left(p + \frac{a}{v^2}\right)(v - b) = c.$$

Draw the graph when $a = .0087$, $b = .0023$, $c = 1.1$. Express dv in terms of dp. What is the meaning of dv/dp?

29. The crushing strength of a hollow cast iron column of length l, inner diameter d, and outer diameter D, is

$$T = 46.65\left(\frac{D^{3.55} - d^{3.55}}{l^{1.7}}\right).$$

Calculate the rate of change of T with respect to D, d, and l, when each of these alone varies.

30. Show that the curve $y = (x - a)^3 + b$ has no maxima or minima.

31. Proceed as in Ex. 30 for $y = (x - a)^5 + b$.

32. Show (see Exs. 10–14, p. 38) that the curve $y = P(x)$ is tangent to the x-axis at points where the polynomial $P(x)$ has a double root.

33. Show that if $P(x)$ is a polynomial, its double roots are also roots of the polynomial $P'(x) = dP(x)/dx$. Hence the H. C. D. of $P(x)$ and $P'(x)$ contains as a factor $x - k$, where k is the double root.

34. Assuming the principle of Ex. 33, find the double roots of each of the following equations:

(*a*) $x^3 + x^2 - 5x + 3 = 0$. (*c*) $x^4 + 2x^3 - 11x^2 - 12x + 36 = 0$.

(*b*) $x^3 + 3x^2 - 4 = 0$. (*d*) $x^4 + 2x^3 - 2x - 1 = 0$.

CHAPTER IV

FIRST APPLICATIONS OF DIFFERENTIATION

PART I. APPLICATIONS TO CURVES—EXTREMES

33. Tangents and Normals. We have seen in § 4, p. 6, that if the equation of a curve C is given in explicit form:

(1) $$y = f(x),$$

the derivative at any point P on C represents the *rate of rise*, or *slope*, of C at P:

(2) $$\left[\frac{dy}{dx}\right]_{\text{at } P} = [\textit{slope of } C]_{\text{at } P} = \textit{slope of } PT = \tan\alpha = [m]_{\text{at } P},$$

where α is the angle XHT, counted from the positive direction of the X-axis to the tangent PT, and where m_P denotes the slope of C at P.

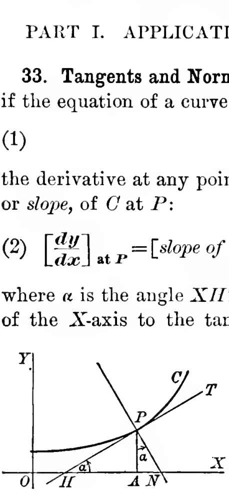


FIG. 13.

Hence (§ 4, p. 7) the *equation of the tangent* is

(3) $$(y - y_P) = \left[\frac{dy}{dx}\right]_P (x - x_P),$$

where the subscript P indicates that the quantity affected is taken with the value which it has at P.

If the slope m_P is *positive*, the curve is *rising* at P; if m_P is *negative*, the curve is *falling*; if m_P is zero, *the tangent is horizontal* (§ 6, p. 8). Points where the slope has any desired value can be found by setting the derivative equal to the given number, and solving the resulting equation for x.

Since, by analytic geometry, the slope n of the *normal* PN is the negative reciprocal of the slope of the tangent, we have,

$$(4) \qquad n_P = \textit{slope of } PN = -\frac{1}{m_P} = -\frac{1}{[dy/dx]_P},$$

as in Ex. 8, p. 11, hence the *equation of the normal* is:

$$(5) \quad (y - y_P) = -\frac{x - x_P}{[dy/dx]_P}, \quad \text{or} \quad (x - x_P) + (y - y_P)\left[\frac{dy}{dx}\right]_P = 0.$$

34. Tangents and Normals for Curves not in Explicit Form. The equation of the curve may not be given in the explicit form (1); instead, it may not be solved for either letter

$$(1) \qquad F(x, y) = 0,$$

as in §§ 26–27, pp. 44–45; or it may be solved for x:

$$(2) \qquad x = \phi(y),$$

as in § 28, p. 46; or the equations in parameter form may be given:

$$(3) \qquad x = f(t), \quad y = \phi(t),$$

as in § 29, p. 47.

In any of these cases, dy/dx can be found by the methods of the articles just cited, and this value may be used in the formulas of § 33. No new formulas are necessary.

In the particular case of the parameter form (3), however, a special formula is sometimes useful. Since by § 29,

$$(4) \qquad m_P = \left[\frac{dy}{dx}\right]_P = \left[\frac{dy}{dt}\right]_P \div \left[\frac{dx}{dt}\right]_P,$$

the equation of the tangent becomes, after simplification,

$$(5) \qquad (y - y_P)\left[\frac{dx}{dt}\right]_P = (x - x_P)\left[\frac{dy}{dt}\right]_P, \quad \text{or} \quad \frac{[dy/dt]_P}{y - y_P} = \frac{[dx/dt]_P}{x - x_P};$$

and the equation of the normal is

$$(6) \qquad (x - x_P)\left[\frac{dx}{dt}\right]_P + (y - y_P)\left[\frac{dy}{dt}\right]_P = 0.$$

A special formula for equations in the implicit form (1) will be given later (§ 164); just now it would actually be inconvenient.

35. Secondary Quantities. In Fig. 13, § 33, since

$$\tan\alpha\ (= m_P = [dy/dx]_P), \text{ and } AP(= y_P),$$

are supposed to be known, the right triangles HAP and PAN can both be solved by the rules of Trigonometry, and the lengths HA, AN, HP, PN can be found in terms of m_P and y_P:

$$[\text{Subtangent}]_P = HA = AP \div \tan\alpha = y_P \div m_P = [y/m]_P.$$

$$[\text{Subnormal}]_P = AN = AP \cdot \tan\alpha = [y \cdot m]_P, \text{ since } \alpha = \angle APN.$$

$$[\text{Length of tangent}]_P = HP = \sqrt{\overline{AP}^2 + \overline{HA}^2} = \sqrt{y_P^2 + [y/m]_P^2}$$
$$= [y\sqrt{1 + (1/m)^2}]_P$$

$$[\text{Length of normal}]_P = PN = \sqrt{\overline{AP}^2 + \overline{AN}^2} = \sqrt{y_P^2 + (y \cdot m)_P^2}$$
$$= [y\sqrt{1 + m^2}]_P.$$

It is usual to give these lengths the names indicated above; and to calculate the numerical magnitudes of these quantities without regard to signs, unless the contrary is explicitly stated.

36. Illustrative Examples. In this article, a few typical examples are solved.

Example 1. Given the curve $y = x^3 - 12x + 7$ (Ex. 2, p. 25), we have $m = dy/dx = 3x^2 - 12$.

(1) The tangent (T) and the normal (N) at a point where $x = a$ are

$$(T)\quad y - (a^3 - 12a + 7) = (3a^2 - 12)(x - a),$$

$$(N)\quad y - (a^3 - 12a + 7) = \frac{-1}{3a^2 - 12}(x - a);$$

thus, at $x = 3$, the tangent and normal are

$$(T)\quad y + 2 = 15(x - 3), \quad (N)\quad y + 2 = -\tfrac{1}{15}(x - 3).$$

(2) The tangent has a given slope k at points where

$$3x^2 - 12 = k, \text{ i.e. } x = \pm\sqrt{\frac{k + 12}{3}};$$

there are always two points where the slope is the same, if $k > -12$; thus if $k=0$, $x = \pm 2$; if $k = -9$, $x = \pm 1$; if $k = -12$, $x = 0$; if $k < -12$, no real value for x exists (see Fig. 17, p. 77).

(3) The secondary quantities of § 35 may be calculated without using the formulas of § 35. Thus, at the point where $x = 3$, the tangent (T) cuts the x-axis where $x = 47/15$; the normal (N) cuts the x-axis where $x = -27$. If the student will draw a figure showing these points and

lines, he will observe directly that the subtangent is $2/15$, the subnormal 30, the length of the tangent $\sqrt{2^2 + (2/15)^2}$, the length of the normal $\sqrt{30^2 + 2^2}$. These values agree with those given by § 35.

(4) The given curve cuts the curve $y = x^3 - 5$ at a point given by solving the two equations simultaneously; this gives $x = 1$, $y = -4$; at this point the slopes of the two curves are $m_1 = -9$, $m_2 = +3$; hence, by Analytic Geometry, the acute angle between them is given by the formula

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-9 - 3}{1 - 27} = \frac{12}{26} = \frac{6}{13},$$

from which θ can be found by use of a trigonometric table (*Tables*, V, *A*). From a larger table, we find $\theta = 24^\circ\, 47'$.

Example 2. Given the circle $x^2 + y^2 = 1$, we have $m = dy/dx = -x/y$ [see § 26].

(1) The tangent (T) and normal (N) at a point (x_0, y_0) are

$$(T)\ (y - y_0) = -\frac{x_0}{y_0}(x - x_0), \quad (N)\ (y - y_0) = \frac{y_0}{x_0}(x - x_0);$$

or, since $x_0{}^2 + y_0{}^2 = 1$,

$$(T)\ xx_0 + yy_0 = 1, \quad (N)\ yx_0 = y_0 x;$$

thus, at the point $(3/5, 4/5)$, which lies on the circle, we have

$$(T)\ 3x + 4y = 5, \quad (N)\ 3y = 4x.$$

(2) The tangent has a given slope k at points where

$$-\frac{x_0}{y_0} = k, \quad i.e.\ \ x_0 + ky_0 = 0.$$

The coördinates (x_0, y_0) can be found by solving this equation simultaneously with the equation of the circle, or by actually drawing the line $x_0 + ky_0 = 0$. Thus the points where the slope is $+1$ lie on the straight line $x + y = 0$; hence, solving $x + y = 0$ and $x^2 + y^2 = 1$, the coördinates are found to be $x = \pm 1/\sqrt{2}$, $y = \mp 1/\sqrt{2}$; but these points are most readily located in a figure by actually drawing the line $x + y = 0$.

(3) The given circle cuts the parabola $9y = 20x^2$ at the points $(\pm 3/5, 4/5)$; at the point $(3/5, 4/5)$ the slopes of the two curves are $m_1 = -3/4$, $m_2 = 40x/9 = 8/3$; hence the acute angle θ between the two curves at that point is

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{41}{12} = 3.4167, \text{ whence } \theta = 73^\circ\, 41'\, 10''.$$

EXERCISES XIII. — TANGENTS AND NORMALS

1. Find the equations of the tangent and that of the normal, and find the four quantities defined in § 35, for each of the following curves at the point indicated:

(a) $y = x^3 - 12x + 7$; $(1, -4)$.

(b) $y = \dfrac{2x - 1}{3x + 2}$; $(-1, 3)$.

(c) $9x^2 + y^2 = 25$; $(1, 4)$.

(d) $xy + y^2 - 2x = 5$; $(-4, 1)$.

(e) $x = y^3 - 3y^2 + 5$; $(3, 1)$.

(f) $\begin{Bmatrix} x = (1 - t)^2 \\ y = (1 + t)^2 \end{Bmatrix}$; $(1, 1)$.

(g) $\begin{Bmatrix} x = t^2 + 4t - 1 \\ y = t^3 - 3t + 5 \end{Bmatrix}$; $(t = 1)$.

2. Find the angle between the curves $y = x^2$ and $y^2 = x$ at each of their common points. See *Tables*, III, A.

3. Find the points (if any) at which each of the curves in Ex. 1 has the slope zero; the slope $+1$.

4. Determine the values of x for which the slope, in each of the curves in Ex. 1, is positive; and those for which it is negative.

5. In Ex. 1, the curves (a) and (c) pass through the point $(1, -4)$; at what angle do they cross?

6. The curves $y = x$, $y = x^2$, $y = x^3$, $\cdots$, $y = x^{-1}$, $y = x^{-2}$, $\cdots$, $y = x^{1/2}$, $y = x^{1/3}$, $\cdots$ all pass through the point $(1, 1)$. Determine the angle which each of these curves makes with the first one of them at that point.

7. Determine the angle between the curves $y = x^n$ and $y = x^m$ at the point $(1, 1)$ where m and n have any values whatever; at the point $(0, 0)$ (only if both n and m are positive). (Special case: $n = p/q$, $m = q/p$, where p and q are integers.)

8. Determine the equation of the tangent and that of the normal to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at any point (x_0, y_0) on it.

[Solution: $2b^2x\,dx + 2a^2y\,dy = 0$, hence $dy/dx = -b^2x/a^2y$; the tangent and normal are, respectively,

$$(T)\ (y - y_0) = -\frac{b^2x_0}{a^2y_0}(x - x_0),\quad (N)\ (y - y_0) = \frac{a^2y_0}{b^2x_0}(x - x_0),$$

or $\quad (T)\ b^2xx_0 + a^2yy_0 = a^2b^2,\quad (N)\ b^2x_0y - a^2xy_0 = x_0y_0(b^2 - a^2),$

since $b^2x_0^2 + a^2y_0^2 = a^2b^2$.]

9. Determine the equation of the tangent and that of the normal to each of the following curves at any point (x_0, y_0) on it:

(a) $y = kx^2$.

(b) $y^2 = 2px$.

(c) $x^2 + y^2 = a^2$.

(d) $y = kx^3$.

(e) $b^2x^2 - a^2y^2 = a^2b^2$.

(f) $ax^2 + 2bxy + cy^2 = f$.

(g) $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$.

(h) $y = (ax + b)/(cx + d)$.

10. The curve whose equations in parameter form are

(1) $$x = 3t + 4,\quad y = t^2 + 2,$$

gives (Example 2, p. 37):

$$m = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2t}{3};$$

hence this curve has a slope 1 when $t = 3/2$, *i.e.* when $x = 17/2$, $y = 17/4$. Its slope is 0 when $t = 0$, *i.e.* at $(4, 2)$.

Verify these facts by drawing an accurate figure; also by eliminating t in (1) and finding the derivative from the explicit equation.

11. Show that the slope of the curve $x^3 + y^3 - 3xy = 0$ (Example 1, p. 46) is $+1$ at points where it cuts the circle $x^2 + y^2 - x - y = 0$. Show that its slope is zero (tangent horizontal) where it cuts the parabola $y = x^2$; that the tangent is vertical $(1/m = 0)$ where it cuts the parabola $y^2 = x$.

12. Draw the curve of Ex. 11 by using its equations in parameter form (Ex. 6 *d*, p. 48):

$$x = \frac{3t}{1 + t^3},\quad y = \frac{3t^2}{1 + t^3},$$

and show that $dy/dx = (2t - t^4)/(1 - 2t^3)$, found from these equations, agrees with the value found from the implicit equation.

37. Extremes.

In § 6, p. 8, and in numerous examples, we have found maxima and minima of functions by first finding the points at which the tangent is horizontal, and then testing these values.

The value of $f(x)$ at a point where $x = a$ is $f(a)$. This value is a $\left\{\begin{matrix}\textbf{maximum}\\ \textbf{minimum}\end{matrix}\right\}$ value if it is $\left\{\begin{matrix}\textbf{greater than}\\ \textbf{less than}\end{matrix}\right\}$ any other value of $f(x)$ for values of x *sufficiently near* to $x = a$.

A maximum or a minimum is called an **extreme value**, or an **extreme** of $f(x)$.

38. Critical Values.

We have seen that a horizontal tangent (*i.e.* slope zero) does not always give rise to an extreme. Thus, the curve $y = x^3$ (Ex. 5 (*b*), p. 11) has a horizontal tangent at the origin; but the origin is neither a highest nor a lowest point.

On the other hand, extremes may also occur at points where the derivative has no meaning, or at points where the function becomes meaningless.

Thus, the curve $y = x^{2/3}$ gives $m = 2/(3\,x^{1/3})$; hence m is meaningless when $x = 0$; in fact, the curve has a vertical tangent at that point. It is easy to see that this is, however, the lowest point on the curve.

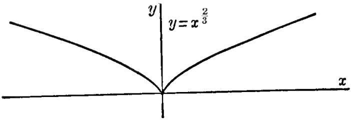


FIG. 14.

Again, if a duplicating apparatus costs \$150, and if the running expenses are 1¢ per sheet, the total cost of printing n sheets is $t = 150 + .01\,n$. This equation represents a straight line; geometrically there are no extreme values of t; but practically t is a minimum when $n = 0$, since negative values of n are meaningless. Such cases are usually easy to observe.

A value of x of any one of the types just mentioned, at which $f(x)$ *may* have an extreme, is called a **critical value**; the corresponding point on the curve $y = f(x)$ is called a **critical point.**

39. Fundamental Theorem. We proceed to show that a function $f(x)$ cannot have an extreme except at a critical point: that is, assuming that $f(x)$ and its derivative have definite meanings at $x = a$ and everywhere near $x = a$, *no extreme can occur if the derivative is not zero at* $x = a$.

We are supposing that all our functions are continuous; if, then, the derivative m is positive at $x = a$, it cannot suddenly become negative or zero. Hence m is positive on both sides of $x = a$, and there can be no extreme there.

Likewise if m is negative, the curve is falling near $x = a$ on both sides of $x = a$; there can be no extreme.

40. Final Tests. It is not certain that $f(x)$ has an extreme value at a critical point. To decide the matter, we proceed to determine whether the curve rises or falls to the left and to

the right of the critical point: it rises if $m > 0$; it falls if $m < 0$.

Near a *maximum*, the curve rises on the left and falls on the right.

Near a *minimum*, the curve falls on the left and rises on the right.

If the curve rises on both sides, or falls on both sides, of the critical point, there is *no extreme* at that point.

41. Illustrative Examples.

Example 1. To find the extremes of the function $y = f(x) = x^3 - 12x + 7$. (See § 6, p. 10.)

(*A*) *To find the Critical Values.* Set the derivative equal to zero and solve for x:

$$m = \frac{dy}{dx} = 3x^2 - 12;\ 3x^2 - 12 = 0;\ x = 2 \text{ or } x = -2.$$

(*B*) *Precautions.* Notice that $f(x)$ and its derivative each has a meaning for every value of x; hence $x = +2$ and $x = -2$ are the only critical values.

(*C*) *Final Tests.* $m = 3x^2 - 12 = 3(x^2 - 4)$ is positive if x is greater than 2, negative if x is slightly less than 2; hence the curve rises on the right and falls on the left of $x = 2$, therefore $f(2) = -9$ is a minimum of $f(x)$. The student may show that $f(-2) = 23$ is a maximum of $f(x)$. (See Fig. 5, p. 10.)

Example 2. To find the extremes of the function

$$y = f(x) = 3x^4 - 12x^3 + 50.$$

(*A*) *Critical Values.* Setting $dy/dx = 0$, and solving, we find:

$$m = \frac{dy}{dx} = 12x^3 - 36x^2;\ 12x^3 - 36x^2 = 0;$$
$$x = 0, \text{ or } x = 3.$$

(*B*) *Precautions.* y and dy/dx have a meaning everywhere; the only critical values are 0 and 3.

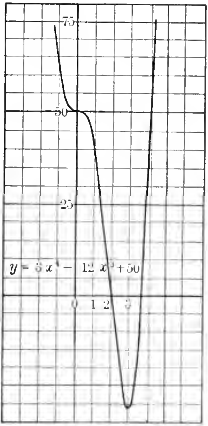


FIG. 15.

(*C*) *Final Tests.* Near $x = 0$, $m = 12x^2(x-3)$ is negative on both sides; hence there is no extreme there, though the tangent is horizontal.

Near $x = 3$, $m = 12x^2(x-3)$ is positive on the right, negative on the left; hence $f(3) = -31$ is a minimum.

The information given above is of great assistance in accurate drawing.

Example 3. Two railroad tracks cross at right angles; on one of them an eastbound train going 15 mi. per hour clears the crossing one minute before the engine of a southbound train running at 20 mi. per hour reaches the crossing. Find when the trains were closest together.

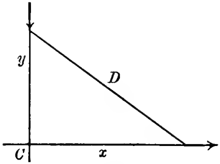


FIG. 16.

Let x and y be the distance in miles of the rear end of the first train and the engine of the second one from the crossing, respectively, at a time t measured in minutes beginning with the instant the first train clears the crossing; then

$$x = \frac{15}{60}t, \quad y = \frac{20}{60}(1-t), \quad D^2 = x^2 + y^2 = \frac{1}{9} - \frac{2}{9}t + \frac{25}{144}t^2$$

where D is the distance between the trains in miles.

Since D is a positive quantity, it is a minimum whenever D^2 is a minimum; hence we write:

$$m = \frac{d(D^2)}{dt} = -\frac{2}{9} + \frac{25}{72}t; \quad -\frac{2}{9} + \frac{25}{72}t = 0; \quad t = \frac{16}{25};$$

when $t < 16/25$, $m < 0$; if $t > 16/25$, $m > 0$; hence D^2 is diminishing before $t = 16/25$ and increasing afterwards. It follows that D is a minimum when $t = 16/25$. Substituting this value for t, we find

$$x = \frac{4}{25}, \quad y = \frac{3}{25}, \quad D^2 = \frac{1}{25};$$

hence the minimum distance between the trains is 1/5 of a mile, and this occurs 16/25 of a minute after the first train clears the crossing.

Example 4. To find the most economical shape for a pan with a square bottom and vertical sides, if it is to hold 4 cu. ft.

Let x be the length of one side of the base, and let h be the height. Let V be the volume and A the total area. Then $V = hx^2 = 4$, whence $h = 4/x^2$; and

$$A = x^2 + 4hx = x^2 + \frac{16}{x};$$

whence we find

$$m = \frac{dA}{dx} = 2x - \frac{16}{x^2};\ 2x - \frac{16}{x^2} = 0,\ x^3 = 8,\ x = 2.$$

When $x < 2$. $m = 2(x^3 - 8)/x^2$ is negative; when $x > 2$, m is positive; hence A is decreasing when x is increasing toward 2, and A is increasing as x is increasing past 2; therefore $x = 2$ gives the minimum total area $A = 12$. Notice that the height is $h = 4/x^2 = 1$. The correct dimensions are $x = 2$, $h = 1$ (in feet).

Example 5. The pan of the preceding example is to sit under a refrigerator which clears the floor by 8 in. How should it be made?

Since h cannot now exceed 8 in. $= 2/3$ ft., it is clear that the minimum of A found in Ex. 4 does not apply. The function $A = x^2 + 16/x$ is meaningless if $h > 2/3$, *i.e.* if $4/x^2 > 2/3$, or $x^2/4 < 3/2$, or $x < \sqrt{6}$ $= 2.45$ (in feet).

Since A is increasing as x increases, x should be made as small as possible; practically, we ought to chose, say $x = 2.5$ ft. $= 30$ in.; then $h = 16/25$ ft. $= 7.68$ in., — we ought to take h about 7 3/4 in., which gives 1/4 in. clearance. This gives $V = 6975$ cu. in., in place of 6912 required, but this difference is on the safe side, and is practically negligible, because it corresponds to a difference in height of much less than 1/8 in.

EXERCISES XIV. — EXTREMES

1. Determine the maximum and minimum values of the following functions and draw the graphs, choosing suitable scales:

(*a*) $y = x^3 - 3x^2 + 1$.

(*b*) $s = 2t^3 - 3t^2 - 36t + 20$.

(*c*) $p = q^3 + 6q^2 - 15q$.

(*d*) $y = x^3 - 2ax^2 + a^2x$.

(*e*) $x = y^4 - 8y^2 + 2$.

(*f*) $v = u^4 - 4u^3 + 4u^2 + 3$.

(*g*) $m = n^5 - 10n^4 + 20n^3 + 32$.

(*h*) $A = r^6 - 6r^4 + 4r^3 + 9r^2 - 12r + 4$.

(*i*) $s = (t-1)(2-t)^2$.

(*j*) $V = h(h-1)^2$.

(*k*) $r = (s^2 - 1)(s^2 - 4)$.

(*l*) $x = (y-2)^3(y+3)^3$.

(*m*) $y = \dfrac{(x+3)^2}{(x+2)^2}$.

(*n*) $v = u + \dfrac{a^2}{u}$.

(*o*) $y = \dfrac{x}{x^2 + 2x + 4}$.

(*p*) $K = \dfrac{h}{ah^2 + bh + c}$.

(*q*) $y = \dfrac{x^3 - x}{x^4 - x^2 + 1}$.

(*r*) $Q = k + \sqrt{1-k}$.

(*s*) $D = r\sqrt{2 - r^2}$.

(*t*) $R = \sqrt[3]{x+6} - x$.

2. What is the largest rectangular area that can be inclosed by a line 100 feet long?

3. What must be the ratio of the sides of a right triangle to make its area a maximum, if the hypothenuse is constant?

4. Determine two possible numbers whose product is a maximum if the sum of their squares is 50. Is there any minimum?

5. Determine two numbers whose product is 100 and such that the sum of their squares is a minimum. Is there any maximum? Did you account for negative possible values of the two numbers?

6. What are the most economical proportions for a cylindrical can? Is there any most extravagant type? Mention other considerations which affect the actual design of a tomato can. Is an ordinary flour barrel this shape? What different considerations enter in making a barrel?

7. What are the most economical proportions for a cylindrical pint cup? (1 pint $= 28\frac{7}{8}$ cu. in.) Mention considerations of practical design.

8. Determine the best proportions for a square tank with vertical sides, without a top. Is there any most extravagant shape?

9. The strength of a rectangular beam varies as the product of the breadth by the square of the depth. What is the form of the strongest beam that can be cut from a given circular log? Mention some other practical considerations which affect actual sawing of timber.

10. The stiffness of a rectangular beam varies as the product of the breadth by the cube of the depth. What are the dimensions of the stiffest beam that can be cut from a circular log?

11. Is a beam of the commercial size $3'' \times 8''$ stronger (or stiffer) than the size $2'' \times 12''$ (1) when on edge, (2) when lying flat?

[Commercial sizes of lumber are always a little short.]

12. What line through the point (3, 4) will form the smallest triangle with the coördinate axes? Is there any other minimum? Any maximum?

13. Determine the shortest distance from the point (0, 3) to a point on the hyperbola $x^2 - y^2 = 16$. Show that it is measured on the normal.

[HINT. Use the square of the distance.]

14. The distance D from the point (2, 0) to the circle $x^2 + y^2 = 1$ is given by the equation $D^2 = 5 - 4x$. Discover the maximum and minimum values of D^2, and show why the ordinary rule fails.

15. Show that the maximum and minimum on the cubic $y = x^3 - ax + b$ are at equal distances from the y axis. Compute y at these points.

16. Show that the cubic $x^3 - ax + b = 0$ has three real roots if the extreme values of the left-hand side (Ex. 15) have different signs. Express this condition algebraically by an inequality which states that the *product* of the two extreme values is negative.

[Any cubic can be reduced to this form by the substitution $x = x' + k$; hence this test may be applied to any cubic.]

17. Show that if the equation $x^3 - ax + b = 0$ has two real roots, the derivative of the left-hand side (*i.e.* $3x^2 - a$) must vanish somewhere between the two roots. Show that the converse is not true.

18. The line $y = mx$ passes through the origin for any value of m. The points (1, 2.4), (3, 7.6), (10, 25) do *not* lie on any one such line: the values of y found from the equation $y = mx$ at $x = 1, 3, 10$ are m, $3m$, $10m$; the differences between these and the given values of y are $(m - 2.4)$, $(3m - 7.6)$, $(10m - 25)$. It is usual to assume that that line for which *the sum of the squares of these differences*

$$S = (m - 2.4)^2 + (3m - 7.6)^2 + (10m - 25)^2$$

is least is the best compromise. Show that this would give $m = 2.50$ (nearly). Draw the figure.

19. In an experiment on an iron rod the amount of stretching s (in thousandths of an inch) and the pull p (in hundreds of pounds) were found to be $(p = 5,\ s = 4)$, $(p = 10,\ s = 8)$, $(p = 20,\ s = 17)$. Find the best compromise value for m in the equation $s = m \cdot p$, under the assumption of Ex. 18. *Ans.* About 5/6.

20. A city's bids for laying cement sidewalks of uniform width and specifications are as follows: Job No. 1: length = 250 ft., cost, \$110; Job No. 2: length, 600 ft., cost, \$250; Job No. 3: 1500 ft., cost, \$630. Find the price per foot for such walks, under the assumption of Ex. 18. How much does this differ from the arithmetic average of the price per foot in the three separate jobs?

21. The amount of water in a standpipe reaches 2000 gal. in 250 sec., 5000 gal. in 610 sec. From this information (which may be slightly faulty) find the rate at which water was flowing into the tank, under assumption of Ex. 18.

22. The values 1 in. = 2.5 cm., 1 ft. = 30.5 cm. are frequently quoted, but they do not agree precisely. The number of centimeters c, and the number of inches i, in a given length are surely connected by an equation of the form $c = ki$. Show that the assumptions of Ex. 18 give $k = 2.541$. Is this the same as the average of the values in the two cases? Which result is more accurate?

23. In experiments on the velocity of sound, it was found that sound travels 1 mi. in 5 sec., 3 mi. in 14.5 sec. These measurements do not agree precisely. Show that the compromise of Ex. 18 gives the velocity of sound 1084 ft. per second. How does this compare with the *average* of the two velocities found in the separate experiments?

24. A quantity of water which at 0° C. occupies a volume v_0, at $\theta°$ C. occupies a volume

$$v = v_0(1 - 10^{-4} \times .5758\,\theta + 10^{-5} \times .756\,\theta^2 - 10^{-7} \times .351\,\theta^3).$$

Show that the volume is least (density greatest), at 4° C. (nearly).

25. Determine the rectangle of greatest perimeter that can be inscribed in a given circle. Is there any minimum?

26. What is the largest rectangle that can be inscribed in an isosceles triangle? Is there any minimum?

27. Find the area of the largest rectangle that can be inscribed in a segment of the parabola $y^2 = 4\,ax$ cut off by the line $x = h$.

28. Determine the cylinder of greatest volume that can be inscribed in a given sphere. Is there also a minimum?

29. Determine the cylinder of greatest convex surface that can be inscribed in a sphere. Is there a minimum?

30. Determine the cylinder of greatest total surface (including the area of the bases) that can be inscribed in a given sphere.

31. What is the volume of the largest cone that can be inscribed in a given sphere?

32. What is the area of the maximum rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$?

PART II. RATES

42. Time-rates. All the applications of derivatives are *rates of increase* (or decrease) of some quantity with respect to some other quantity which is taken as the standard of comparison, or independent variable.

Among all rates, those which occur most frequently are **time-rates**, that is, rate of change of a quantity with respect to the time.

43. Speed. Thus the speed of a moving body is the time-rate of increase of the distance it has traveled:

(1) $$v = \textit{speed}^* = \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt},$$

as in § 7, p. 12, and in numerous examples.

44. Tangential Acceleration. The *speed* itself may change; *the time-rate of change of speed is called the* **acceleration along the path,** *or the* **tangential acceleration.**†

(2) $$j_T = \textit{tangential acceleration}\,\dagger = \lim_{\Delta t \doteq 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}.$$

Thus for a body falling from rest, if g represents the gravitational constant,

$$s = \tfrac{1}{2} gt^2;$$

hence

$$v = \frac{ds}{dt} = gt,$$

and

$$j_T = \frac{dv}{dt} = g;$$

it follows that the tangential acceleration of a body falling from rest is constant; that constant is precisely the gravitational constant g. ‡

In obtaining the tangential acceleration, we actually differentiate the distance s twice, once to get v, and again to get dv/dt or j_T, hence the tangential acceleration is also said to be the **second derivative** of the distance s passed over.

45. Second Derivatives, Flexion. It often happens, as in § 44, that we wish to differentiate a function twice. In any

* The *speed* v is distinguished from the velocity $\mathbf{v}$ by the fact that the speed does not depend on the direction; when we speak of velocity we shall always denote it by $\mathbf{v}$ (in black-faced type) and we shall specify the direction.

† The **general acceleration** $\mathbf{j}$ is also a directed quantity; when we speak of the **acceleration** $\mathbf{j}$ (not tangential acceleration j_T) we shall denote it by $\mathbf{j}$, and give its direction. As in the case of speed, the letter j, in italic type, denotes the value of $\mathbf{j}$ without its direction. (See Ex. 17, p. 74.)

‡ The value of g is approximately 32.2 ft. per second per second = 981 cm per second per second.

case, given $y = f(x)$, the slope of the graph is

$$m = \frac{dy}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}.$$

The slope itself may change (and it always does except on a straight line); *the rate of change of the slope with respect to x will be called the* **flexion** * *of the curve:*

$$b = \textit{flexion} = \frac{dm}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta m}{\Delta x},$$

and will be denoted by b, the initial letter of the word *bend.*

Thus for $y = x^2$, $m = 2x$, $b = 2$ †; for $y = x^3$, $m = 3x^2$, $b = 6x$; for $y = x^3 - 12x + 7$, $m = 3x^2 - 12$, $b = 6x$; for *any* straight line $y = kx + c$, $m = k$, $b = 0$.

The value of b is obtained by differentiating the given function twice; the result is called a **second derivative**, and is represented by the symbol:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dm}{dx} = b.$$

Likewise, the tangential acceleration in a motion is

$$\frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{dv}{dt} = j_T.$$

If the relation between s and t is represented graphically, the *speed* is represented by the *slope*, the *tangential acceleration* by the *flexion*, of the graph. Thus if $s = gt^2/2$ be represented graphically, as in Fig. 6, p. 13, the *slope* of the graph is

$$m = \textit{slope} = \frac{ds}{dt} = gt = \textit{speed} = v,$$

and the flexion of the graph is

$$b = \textit{flexion} = \frac{dm}{dt} = \frac{d^2s}{dt^2} = \frac{dv}{dt} = g = \textit{tangential acceleration} = j_T.$$

* The word *curvature* is used in a somewhat different sense. See § 97, p. 169.

† The flexion for this parabola is constant; note that this means the rate of change of m per unit increase in x, not per unit increase in length along the curve. See § 61, p. 106.

EXERCISES XV. SECOND DERIVATIVES — ACCELERATION

[In addition to this list, the second derivatives of some of the functions in the preceding exercises may be calculated.]

1. Calculate the first and second derivatives in the following exercises. Interpret these exercises geometrically, and also as problems in motion, with s and t in place of y and x:

(a) $y = x^2 + 3x - 4$.
(b) $y = -x^2 + 3x - 4$.
(c) $y = 2x^2 - x - 15$.
(d) $y = -2x^2 - x - 15$.
(e) $y = x^2 - \frac{5}{2}x - 21$.
(f) $y = x^3 - 3x^2 + 1$.
(g) $y = 2x^3 - 3x^2 - 36x - 20$.
(h) $y = x^4 - 8x^2 + 2$.
(i) $y = x^4 - 2x^3 + 5x^2 + 2$.
(j) $y = (1 + x) \div (1 - x)$.
(k) $y = \sqrt{x} + \sqrt{x^2 + 1}$.
(l) $y = (2 - 3x)^2 (3 + x)$.
(m) $y = (x + 2)^3 (x^2 - 1)$.
(n) $y = \sqrt{1 + x} \div \sqrt{1 - x}$.
(o) $y = ax + b$.
(p) $y = c$ (a constant).
(q) $y = ax^2 + bx + c$.
(r) $y = c(x - a)^n$.
(s) $y = (x - a)^m (x - b)^n$.
(t) $y = Ax^{-k}$.

2. Show that the flexion of a straight line is everywhere zero.

3. Show that if the distance passed over by a body is proportional to the time the tangential acceleration is zero. What is the speed in this case?

4. Show that the flexion of the curve $y = ax^2 + bx + c$ is everywhere the same, and equal to twice the coefficient of x^2.

5. Show that if the space-time equation is $s = at^2 + bt + c$, the acceleration is always the same and equal to twice the coefficient of t^2. Is such a motion at all liable to occur in nature?

6. Find the flexion of the curve $y = 1/x$. Show that it resembles y itself in some ways. Does the slope also resemble y? Which one resembles y the more closely?

7. Can you interpret Ex. 6 as a motion problem? What is true at the *beginning* of the motion ($t = 0$)? Can a curve with a vertical asymptote represent a motion? Can a *piece* of such a curve?

8. Find the flexion of the curve $y = (x - 2)^3 (x + 3)^2 (x - 4)$. Show that the flexion has a factor $(x - 2)$, while the slope has a factor $(x - 2)^2 (x + 3)$.

9. Show that the flexion of the curve $y=(x-a)^3(x^2+5)$ has a factor $(x-a)$.

10. If the function $y=x^5+ax^4+bx^3+cx^2+dx+e=0$ has a factor $(x-\alpha)^3$, show that dy/dx has a factor $(x-\alpha)^2$, and d^2y/dx^2 has a factor $(x-\alpha)$.

11. If the equation $x^5+ax^4+bx^3+cx^2+dx+e=0$ has a triple root $x=\alpha$, show that the equation $20\,x^3+12\,ax^2+6\,bx+2\,c=0$ has a factor $x-\alpha$.

12. Show how to find the double and triple roots of any algebraic equation by the Highest Common Divisor process.

13. If the equations of the curve in parameter form are $x=t^3$, $y=t^2$, find the slope m and the flexion b in terms of t.

[HINT. First find m; then use the values of m and x in terms of t to find dm/dx.]

14. Find m and b for each of the following parameter forms:

(a) $x=a+bt$, $y=c+dt$.

(b) $x=t^2$, $y=t^3$. [See Ex. 13.]

(c) $x=t$, $y=t^{-2}$; $t=1$ and 2.

(d) $x=1+t$, $y=\dfrac{1}{1-t}$; $t=\pm 2$.

(e) $x=\dfrac{t}{1+t}$, $y=\dfrac{1-t}{t}$; $t=1$.

(f) $x=\dfrac{3\,t}{1+t^3}$, $y=\dfrac{3\,t^2}{1+t^3}$; $t=1$.

15. If the equations of Ex. 13 express the position of a moving particle at the time t, find the horizontal speed $v_x=dx/dt$ and the vertical speed $v_y=dy/dt$. A second differentiation gives the time-rates of change of these component speeds: $j_x=dv_x/dt=d^2x/dt^2$ and $j_y=dv_y/dt=d^2y/dt^2$. Find each of these quantities in Ex. 13. In each of the exercises in Ex. 14.

16. The **total speed** $v=\sqrt{v_x^2+v_y^2}$ can be found as in Ex. 7, p. 49, from the values of v_x and v_y. Find v in each of the examples of Exs. 13 and 14.

17. The **component accelerations** j_x and j_y of Ex. 15 may be combined to get the **total acceleration** $j=\sqrt{j_x^2+j_y^2}$ by the so-called parallelogram law of physics. Find j in each of the examples of Exs. 13 and 14.

18. The **tangential acceleration** j_T can be found directly from Ex. 16, by means of its definition $j_T=dv/dt$. Find j_T in Exs. 13 and 14. *Show that j_T and j are different in every exercise except* 14 (a).

[The reason for this difference is not difficult: j_T is the acceleration *in the path itself*; j is the *total* acceleration, part of its effect being precisely to

make the path curved; hence a part of j is expended not to *increase the speed*, but to change the *direction* of the speed, *i.e.* to *bend the path*. Notice that Ex. 14 (*a*) represents a straight line path; on it $j_T = j$; this holds *only* on straight line paths. In uniform motion on a circle, for example, $j_T = 0$.]

46. Concavity. Points of Inflexion. If the flexion $b = dm/dx$ is positive, the slope is increasing, and the curve turns upwards, or is *concave upwards*; if the flexion is negative, the slope is decreasing, and the curve is *concave downwards*.

Thus $y = x^2$ is concave upwards everywhere, since $b = 2$ is positive. For $y = x^3$ we find $b = 6x$, which is positive when x is positive, and negative when x is negative; hence $y = x^3$ is concave upwards at the right, and concave downwards at the left of the origin.

A point at which the curve changes from being concave upwards to being concave downwards, or conversely, is called a **point of inflexion.**

The value of the flexion b changes from positive to negative, or conversely, in passing such a point; hence *the value of b at a point of inflexion is zero*, if it has any value there.*

Thus the origin is a point of inflexion on the curve $y = x^3$, for the curve is concave downwards on the left, concave upwards on the right, of the origin.

47. Second Test for Extremes. In seeking the extreme values of a function $y = f(x)$, we find first the *critical points* (§ 38, p. 63), *i.e.* the points at which the tangent is horizontal.

If, at a critical point, $b = d^2y/dx^2 > 0$, the curve is also *concave upwards*,† and the function has a *minimum* there; if $b < 0$, the curve is *concave downwards*, and $f(x)$ has a *maximum*; that is,

$$\text{if } m = \frac{dy}{dx} = 0 \text{ and } b = \frac{d^2y}{dx^2}\begin{Bmatrix} > 0 \\ < 0 \end{Bmatrix} \text{ at } x = a,\ f(a) \text{ is a } \begin{Bmatrix} \textit{minimum} \\ \textit{maximum} \end{Bmatrix}.$$

* Points where the tangent is vertical, for example, may be points of inflexion.

† The curve is then also concave upwards on both sides of the point; if the curve is concave upwards on one side and downwards on the other, b must be zero if it exists at the point.

Whenever the flexion is not zero at a critical point, this method usually furnishes an easy final test for extremes. If the flexion is zero, no conclusion can be drawn directly by this method.* (See, however, § 135.)

48. Illustrative Examples.

Example 1. Consider the function $y = x^3 - 12x + 7$. See Ex. 3, p. 10, and Ex. 1, p. 65. The slope and the flexion are, respectively,

$$m = \frac{dy}{dx} = 3x^2 - 12, \quad b = \frac{d^2y}{dx^2} = \frac{dm}{dx} = 6x.$$

The critical points are (see Ex. 1, p. 65) $x = \pm 2$. Since $6x$ is positive when x is positive, b is positive for $x > 0$; likewise $b < 0$ when $x < 0$. Hence the curve is concave upwards when $x > 0$, and concave downwards when $x < 0$. At $x = +2$, $b > 0$, hence by § 47, y has a minimum at $x = +2$; at $x = -2$, $b < 0$, hence y has a maximum (compare p. 10 and p. 65).

To find a point of inflexion first set $b = 0$;

$$b = \frac{dm}{dx} = \frac{d^2y}{dx^2} = 6x = 0, \quad i.e. \quad x = 0.$$

Since dm/dx is negative for $x < 0$ and positive for $x > 0$, the given curve is concave downwards on the left and concave upwards on the right of this point; hence $x = 0$, $y = 7$ is a point of inflexion. (See Fig. 17, and § 49, p. 77.)

Example 2. Consider the function $y = 3x^4 - 12x^3 + 50$ (Ex. 2, p. 65). The slope and the flexion are, respectively,

$$m = \frac{dy}{dx} = 12x^3 - 36x^2; \quad b = \frac{dm}{dx} = \frac{d^2y}{dx^2} = 36x^2 - 72x.$$

The critical points are $x = 0$, $x = 3$. At $x = 3$, $b = 108 > 0$, hence y is a minimum there. At $x = 0$, $b = 0$, and no conclusion is reached by this method (compare, however, p. 65). To find points of inflexion, first set $b = 0$;

$$b = \frac{dm}{dx} = \frac{d^2y}{dx^2} = 36x^2 - 72x = 0, \; i.e. \; x = 0 \text{ or } x = 2.$$

* Even in this case one *may* decide by determining whether the curve is concave upwards or downwards on both sides of the point; but the method of § 40 is usually superior.

Near $x = 0$, at the left, $dm/dx = 36\,x\,(x - 2)$ is positive, at the right, negative; the given curve is concave upwards on the left, downwards on the right, and ($x = 2$, $y = 2$) is a point of inflexion. (See Fig. 15, and § 49.)

Example 3. For a body thrown vertically upwards, the distance s from the earth is:

$$s = -\tfrac{1}{2}\,gt^2 + v_0 t,$$

where v_0 is the speed with which it is thrown.

The speed and the tangential acceleration are, respectively,

$$v = \frac{ds}{dt} = -\,gt + v_0\,; \quad j_T = \frac{d^2s}{dt^2} = \frac{dv}{dt} = -\,g.$$

If we draw a graph of the values of s and t, the speed v (slope of the graph) is zero when

$$v = -\,gt + v_0 = 0, \quad i.e.\ t = v_0/g,$$

that is, the point is a critical point on the graph. The tangential acceleration (flexion of the graph) is negative everywhere, hence the graph is *concave downwards.*

In particular at the critical point just found, b is negative; hence s has a maximum there:

$$s = -\frac{1}{2}\,gt^2 + v_0 t = \frac{1}{2}\frac{v_0^2}{g}, \quad \text{when } t = \frac{v_0}{g}.$$

The figure, for the special values $v_0 = 64$ and $g = 32$, is drawn in § 49.

49. Derived Curves. It is very instructive to draw in the same figure graphs which give the values of the original function, its derivative, and its second derivative.

These graphs of the derivatives are called the **derived curves**; they represent the *slope* (or *speed* in case of a motion) and the *flexion* (or *tangential acceleration*).

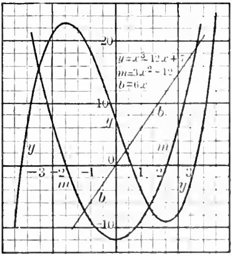
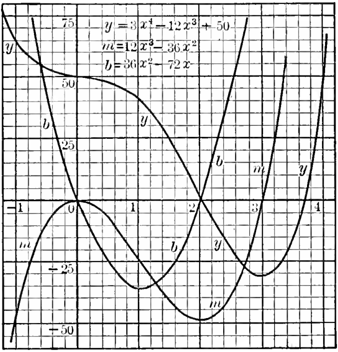


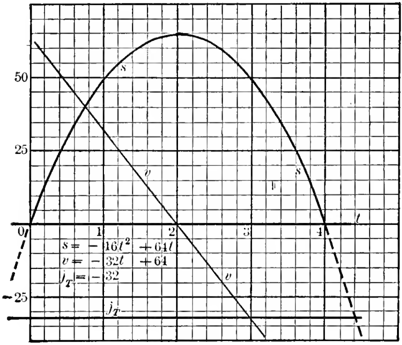
Fig. 17.

The figures for the curves of Exs. 1 and 2 of § 48 are appended. The student should show that each statement made in § 48 and each statement made



$y = 3x^4 - 12x^3 + 50$
$m = 12x^3 - 36x^2$
$b = 36x^2 - 72x$
75
50
25
-25
-50
-1
0
1
2
3
4
y
b
m

FIG. 18.



50
25
-25
0
1
2
3
4
t
s
v
$s = -16t^2 + 64t$
$v = -32t + 64$
$j_T = -32$
j_T

FIG. 19.

on p. 65, for each of the examples, is illustrated and verified in these figures.

The similar curves for space, speed, and acceleration are drawn in Fig. 19, for the motion of a body thrown upwards:

$$s = -\tfrac{1}{2}\,gt^2 + v_0 t \text{ for } g = 32,\ v_0 = 64.$$

Verify the statements made in Ex. 3, § 48.

In drawing such curves, the second derivative should be drawn first of all; the information it gives should be used in drawing the graph of the first derivative, which in turn should be used in drawing the graph of the original function.

EXERCISES XVI. — FLEXION — DERIVED CURVES

1. Draw, in the order just indicated, the first and second derived curves in Ex. 1 (*a*), List XIV, p. 67; and show that each step of your work in that example is exhibited by these figures.

2. Draw the derived curves for Exs. 1 (*b*), 1 (*d*), 1 (*f*), 1 (*n*) of List XIV, p. 67; and show their connection with your previous work.

3. Draw the original and the derived curves for the function $y = x^3 - 9x^2 + 15x - 6$. Find the extreme values of y, and explain the figures. For what value of x is the flexion zero? Does this give a point of inflexion on the original curve?

4. Find the extreme values of y and the points of inflexion on the following curves; in each case draw complete figures:

(*a*) $y = 2x^3 - 3x^2 - 36x$.	(*f*) $y = Ax^2 + Bx + C$.
(*b*) $y = 4x^3 - x^2 - 24x$.	(*g*) $y = mx + n$.
(*c*) $y = x^3 + x^2$.	(*h*) $y = \sqrt{x}$.
(*d*) $y = x^4 - 2x^2 + 40$.	(*i*) $y = x^3 + px + q$.
(*e*) $y = x(x+2)^3$.	(*j*) $y = x^2 + 16/x$.

5. Show that the flexion of the hyperbola $xy = a^2$ varies inversely as the cube of the abscissa x.

6. Show that the flexion of the conic $Ax^2 + By^2 = 1$ (ellipse or hyperbola) varies inversely as the cube of the ordinate y.

7. What is the effect upon the flexion of changing the sign of a in the equation $y = ax^2 + bx + c$?

8. A beam of uniform depth is said to be of "uniform strength" (in resisting a given load) if the actual shape of its upper surface under the load is of the form $y = ax^2 + bx + c$, where x and y represent horizontal

and vertical distances measured from the middle point of the beam's surface in its original (unbent) position. Show that the flexion of such a beam is constant.

9. Show that the addition of a constant to the value of y does not affect the slope nor the flexion.

10. Show that the addition of a term of the form $kx + c$ to the value of y does not affect the flexion. What effect does it have upon the slope?

11. Show, by means of Exs. 9 and 10, that any beam in which the flexion is constant has the form specified in Ex. 8.

12. Show, by a process precisely similar to that of Ex. 11, that a motion in which the tangential acceleration is constant is defined by an equation of the form $s = at^2 + bt + c$.

13. What is the effect upon the graph of an equation if a constant is added to y? How are the positions of the maxima and minima affected? [Take into account vertical as well as horizontal displacement.]

14. What is the effect upon the points of inflexion if a term $kx + c$ is added to the value of y? Will this change in the original curve change the values of x which correspond to extreme values of y?

15. Show that the curve $(1 + x^2)y = (1 - x)$ has three points of inflexion which lie on a straight line.

16. Show that the graph of a polynomial of the n^{th} degree cannot have more than $n - 2$ points of inflexion.

17. Show that if a polynomial has a factor $(x-a)^k$, its flexion has a factor $(x - a)^{k-2}$.

18. Find, by the methods of Exs. 9–12, what the form of y must be if the slope is:

$$(a)\ \frac{dy}{dx} = 0;\quad (b)\ \frac{dy}{dx} = -3;\quad (c)\ \frac{dy}{dx} = 6x;\quad (d)\ \frac{dy}{dx} = ax + b.$$

19. What is the form of y if the flexion is 6? if the flexion is $2x + 3$? if the flexion is zero?

20. If a beam of length l is supported only at both ends, and loaded by a weight at its middle point, its deflection y at a distance x from one end is $y = k\,(3l^2x - 4x^3)$, provided the cross section of the beam is constant. Find the flexion and show that there are no points of inflexion between the supports.

21. If the beam of Ex. 20 is rigidly fixed at both ends, and loaded at its middle point, the deflection of each half of the beam is $y = k\,(3lx^2 - 4x^3)$, where x is measured from either end. Show that there is a point of

inflexion at a distance $l/4$ from the end, and that the greatest deflection is at the middle point.

22. Find the points of inflexion and the point of maximum deflection of a uniform beam of length l whose deflection is:

$$(a)\quad y = k\,(3\,lx^2 - x^3).$$

[Beam rigidly embedded at one end, loaded at other end. Origin at fixed end.]

$$(b)\quad y = k\,(3\,x^2l^2 - 2\,x^4).$$

[Beam freely supported at both ends, loaded uniformly. Origin at lowest point.]

$$(c)\quad y = k\,(6\,l^2x^2 - 4\,lx^3 + x^4).$$

[Beam embedded at one end only; loaded uniformly. Origin at fixed end.]

$$(d)\quad y = k\,(l^3x - 3\,lx^3 + 2\,x^4).$$

[Beam embedded at one end, supported at the other end; loaded uniformly. Origin at free end.]

50. Angular Speed. If a wheel turns, a given spoke of it makes an angle θ with its original position which changes with the time, *i.e.* θ is a function of the time:

$$\theta = f(t).$$

The time-rate of change of the angle θ is called the **angular speed**; *it is denoted by* ω:

$$\omega = \textit{angular speed} = \frac{d\theta}{dt} = \lim_{\Delta t \doteq 0} \frac{\Delta\theta}{\Delta t}.$$

51. Angular Acceleration. The angular speed may change; *the time-rate of change of the angular speed is called the* **angular acceleration**; *it is denoted by* α:

$$\alpha = \textit{angular acceleration} = \lim_{\Delta t \doteq 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

Example 1. A flywheel of an engine starts from rest, and moves for 30 seconds according to the law

$$\theta = -\frac{1}{1800}t^4 + \frac{1}{30}t^3,$$

where θ is measured in degrees, after which it rotates uniformly.

Then

$$\omega = \frac{d\theta}{dt} = -\frac{1}{450}t^3 + \frac{1}{10}t^2,$$

and

$$\alpha = \frac{d\omega}{dt} = -\frac{1}{150}t^2 + \frac{2}{10}t.$$

This example furnishes an instance in which *the derived curves, i.e.,* the graphs which show the values of ω and of α *are more important than the original curve;* for the total angle described is relatively unimportant.

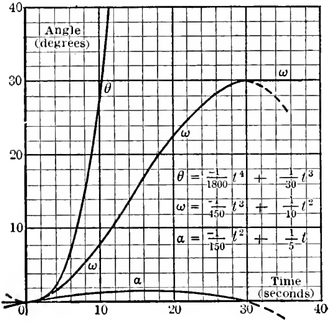


FIG. 20.

In the figure a scale is chosen which shows particularly well the variation of ω; θ is allowed to run off of the figure completely, since its values are uninteresting.

The acceleration α is so arranged that it does not suddenly drop to zero when the flywheel is allowed to run uniformly; and the values of α are never large. Something resembling this figure is what actually occurs in starting a large flywheel.

In actual practice with various machines, curves of this type are often drawn *experimentally;* the equations serve only as approximations to the reality; but they are often indispensable in calculating other related quantities, such as the acceleration in this example.

Curves which resemble the graph of ω in this example occur frequently. (See §§ 87, 134.)

52. Momentum. Force. As a further illustration of time-rates, we mention a statement often given as the definition of **force**: *force is the time-rate of change of momentum.* (Compare Newton's Second Law of Motion.)

The momentum M of a body moving in a straight path is defined as the product of the mass m of the body times its speed v:

$$M = m \cdot v.$$

The force acting on the body is therefore

$$F = \frac{dM}{dt} = \lim_{\Delta t \doteq 0} \frac{\Delta M}{\Delta t} = \frac{d(m \cdot v)}{dt} = m \cdot \frac{dv}{dt} = m \cdot j_T = m \frac{d^2 s}{dt^2}.$$

This law is often stated in the form: *the force is the product of the mass times the acceleration;* for the present the results are stated only for a body moving in a straight line along which the force itself acts.

This consideration of time-rates makes clear that the two definitions of force quoted above are equivalent.

EXERCISES XVII. — TIME-RATES

1. Express as a time-rate the speed v of a moving body, and write the result as a derivative.

2. Express as time-rates the following concepts:
   (*a*) The tangential acceleration of a moving body.
   (*b*) The horizontal speed of a moving body.
   (*c*) The vertical speed of a moving body.
   (*d*) The speed of evaporation of a liquid exposed to air.
   (*e*) The speed of formation of rust on iron.
   (*f*) The rate of growth of the height of a tree.
   (*g*) The rate of fluctuation of the value of gold.
   (*h*) The rate of rise or fall of the height of a river.

3. In Ex. 2, which of the rates mentioned are surely constant; which may possibly be constant in some instances; which may be constant part of the time? For which of them does a concept analogous to acceleration have a meaning?

4. If such a rate is constant, how can the total amount (or value) of the changing quantity be computed? Find the total amount of water in a tank which originally contained 2000 gal., after water has run into it for 10 min. at the rate of 10 gal. a second.

5. If a train, after it is 10 mi. from Chicago, travels directly away at 60 mi. an hour, how far is the train from Chicago 5 min. later?

6. If y is any varying quantity, and if $dy/dt = 7$, express y in terms of t if $y = 10$ when $t = 0$. Again, if $y = 5$ when $t = 0$.

7. If $dy/dt = 2t + 3$, express y in terms of t if $y = 0$ when $t = 0$. [See Exs. 9, 10, List XVI.]

8. A flywheel rotates so that $\theta = t^3 \div 1000$, where θ is the angle of rotation (in degrees) and t is the time (in seconds). Calculate the angular speed and acceleration, and draw a figure to represent each of them.

9. Suppose that a wheel rotates so that $\theta = t^3 \div 1000$ where θ is measured in *radians* [1 radian $= 180°/\pi$]. Is its speed greater than or less than that of the wheel in Ex. 8? What is the ratio of the speeds in the two cases?

10. Compare linear speeds in miles per hour with speeds in feet per minute. Reduce 60 miles per hour to feet per second.

11. Compare angular speeds in radians per second to speeds in degrees per second. Reduce 90° per second to radians per second.

12. Compare angular speeds in revolutions per minute (R. P. M.) with speeds in degrees per second. Express the angular speed in Example 1, § 51, in R. P. M.

13. Reduce a linear acceleration 60 in./sec./sec. to ft./sec./sec.; to in./min./min.; to ft./min./min. Express the acceleration due to gravity ($g = 32.2$ ft./sec./sec.) in each of these units.

14. Reduce the angular acceleration in Example 1, § 51, to rev./sec./sec.; to rev./min./min.

15. If a wheel moves so that $\theta = -t^4/16 - t/32$, where θ is measured in radians and t in minutes, find the angular speed and acceleration in terms of radians and minutes; in terms of revolutions and minutes; in terms of radians and seconds (of time).

16. If a Ferris wheel turns so that $\theta = 20\,t^2$ while changing from rest to full speed, where θ is in degrees and t in minutes, when will the speed reach 20 revolutions per hour?

17. If the angular speed is $\omega = kt$ as in Ex. 16, show that the acceleration α is constant. Conversely, show that if $\alpha = k$, and if t is the time since starting, $\omega = kt$.

18. How far does a point on the rim of a wheel travel during one complete revolution? Express the linear speed of a point on the rim of a wheel 10 ft. in diameter when the angular speed is 4 R. P. M.

19. Express in miles per hour the speed of a point of a wheel 2 ft. in diameter which is rotating with an angular speed of 10 revolutions per second.

20. If the Ferris wheel of Ex. 16 is 100 ft. in diameter, what is the linear speed of the rim at 20 R. per hour?

21. Find the linear speed and the tangential acceleration of a point on the rim of the wheel of Ex. 1, § 51, if the wheel is 10 ft. in diameter. What are they when $t = 30$ sec.?

22. Find the linear speed and acceleration in Ex. 8, if the radius of the wheel is 4 ft. How large would the wheel of Ex. 9 have to be to make the linear speed of its rim the same?

23. An engine with driving wheels 5 ft. in diameter is traveling 40 mi./hr. Express the angular speed of the rim in revolution per minute.

24. If a train starts from a station with speed $v = t/2 + t^2/100$ (in feet and seconds), find the angular speed and hence the angular acceleration of drivers 6 ft. in diameter. What is the value of each of these quantities when $t = 10$?

25. Find the momentum ($=$ mass $\times$ speed) of a falling body, if the distance passed over is $s = gt^2/2$. Find the force acting.

[NOTE. If *force* is measured in pounds, *mass* $=$ weight in pounds $\div\, g$. Hence, force $=$ mass $\cdot\, g$.]

26. If a body moves so that $s = 3t^2 - 12$, find the force acting if the body weighs 10 lb.

27. The hammer of a pile driver weighs 1000 lb. If it drops 15 ft. onto a pile according to the law of Ex. 25, what is the momentum of its impact? The average force of the blow is the average rate at which the momentum is destroyed. How much is this if the hammer is stopped in 1/1000 sec.?

28. What is the average force of a hammer blow by a 2-lb. hammer moving at 30 ft./sec., stopped in 1/1000 sec.?

29. The kinetic energy of a moving body is $E = mv^2/2$. Show that $dE/dt = mv \cdot dv/dt =$ momentum $\times$ acceleration.

30. An electric current c (measured in amperes) is the quantity q of electricity (in coulombs) which passes a given point per second. Express this fact in the derivative notation.

53. Related Rates. If a relation between two quantities is known, the time-rate of change of one of them can be expressed in terms of the time-rate of change of the other.

Thus, in a spreading circular wave caused by throwing a stone into a still pond, the circumference of the wave is

$$c = 2\pi r, \qquad (1)$$

where r is the radius of the circle. Hence

$$\frac{dc}{dt} = 2\pi\frac{dr}{dt}; \tag{2}$$

or, the time-rate at which the circumference is increasing is 2π times the time-rate at which the radius is increasing. (Compare Ex. 8, p. 27.) Dividing both sides by dr/dt, we find

$$\frac{dc}{dt} \div \frac{dr}{dt} = 2\pi = \frac{dc}{dr} = dc \div dr;$$

that is, *the ratio of the time-rates is the derivative of* c *with respect to* r; or, *the ratio of the time-rates is equal to the ratio of the differentials.*

The fact just mentioned is true in general; if y and x are any two related variables which change with the time, it is true (Rule [VII$_a$], p. 40) that:

$$\frac{dy}{dt} \div \frac{dx}{dt} = \frac{dy}{dx} = dy \div dx,$$

that is, *the ratio of the time-rates of* y *and* x *is equal to the ratio of their differentials, i.e.* to the *derivative* dy/dx.

Example 1. Water is flowing into a cylindrical tank. Compare the rates of increase of the total volume and the increase in height of the water in the tank, if the radius of the base of the tank is 10 ft. Hence find the rate of inflow which causes a rise of 2 in. per second; and find the increase in height due to an inflow of 10 cu. ft. per second. Consider the same problem for a conical tank.

(A) The volume V is given in terms of the height h by the formula:

$$V = \pi r^2 h = 100\pi h,$$

hence
$$\frac{dV}{dt} = 100\pi\frac{dh}{dt};$$

or, the rate of increase in volume (in cubic feet per second) is 100π times the rate of increase in height (in feet per second).

If $dh/dt = 1/6$ (measured in feet per second), $dv/dt = 100\ \pi/6 =$ (roughly) 52.3 (cubic feet per second). If $dv/dt = 10$, $dh/dt = 10 \div 100\pi$ = (roughly) .031 (in feet per second) = 22.3 (in inches per minute).

(*B*) If the reservoir is *conical*, we have

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi h^3 \tan^2 \alpha,$$

where r is the radius of the water surface, h the height of the water, and α the half-angle of the cone; for $r = h \tan \alpha$. In this case

$$\frac{dV}{dt} = \pi h^2 \tan^2 \alpha \frac{dh}{dt},$$

which varies with h. If $\alpha = 45°$ ($\tan \alpha = 1$), at a height of 10 ft., a rise 1/6 (feet per second) would mean an inflow of $\pi h^2 \times (1/6) = 100\pi/6 = 52.3$ (cubic feet per second). At a height of 15 feet, a rise of 1/6 (feet per second) would mean an inflow of $225\pi/6$ = (roughly) 117.8 (cubic feet per second). An inflow of 100 (cubic feet per second) means a rise in height of $100/\pi h^2$, which varies with the height; at a height of 5 ft., the rate of rise is $4/\pi = 1.28$ (feet/second).

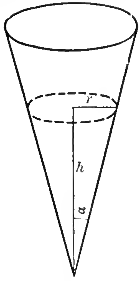


FIG. 21.

Example 2. A body thrown upward at an angle of 45°, with an initial speed of 100 ft. per second, neglecting the air resistance, etc., travels in the parabolic path

$$y = -\frac{gx^2}{10000} + x,$$

where x and y mean the horizontal and vertical distances from the starting point, respectively; g is the gravitational constant $= 32.2$ (about); and the horizontal speed has the constant value $100/\sqrt{2}$. Find the vertical speed at any time t, and find a point where it is zero.

The horizontal speed and the vertical speed, *i.e.* the time-rate of change of x and y, respectively, are connected by the relation (see §§ 8, 29.)

$$\frac{dy}{dt} \div \frac{dx}{dt} = \frac{dy}{dx} = -\frac{gx}{5000} + 1;$$

hence $$\frac{dy}{dt} = \left(-\frac{gx}{5000} + 1\right)\frac{dx}{dt} = -\frac{gx}{50\sqrt{2}} + \frac{100}{\sqrt{2}}.$$

This vertical speed is zero where

$$-\frac{gx}{50\sqrt{2}} + \frac{100}{\sqrt{2}} = 0, \; i.e. \; x = \frac{5000}{g} = 155.3 \text{ (about)},$$

which corresponds to $y = 2500/g = 77.7$ (about). At this point the vertical speed is zero; just before this it is positive, just afterwards it is negative. When $x = 0$ the value of dy/dt is $100/\sqrt{2}$; when $x = 2500/g$, $dy/dt = 50/\sqrt{2}$; when $x = 7500/g$, $dy/dt = -50/\sqrt{2}$.

EXERCISES XVIII. — RELATED RATES

1. Water is flowing into a tank of cylindrical shape at the rate of 50 gal. per minute. If the tank is 8 ft. in diameter, find the rate of increase in the height of the water in the tank.

2. Water is flowing into a cone-shaped tank, 20 ft. across at the bottom and 15 ft. high, at the rate of 100 cu. ft. per minute. Calculate the rate of increase of the water level.

How fast is the water entering the same tank when the height is 6 ft., if the level is rising 6 in. per minute?

3. A funnel 8 in. across the top and 6 in. deep is being emptied at the rate of 2 cu. in. per minute. How fast does the surface of the liquid fall?

4. A hemispherical bowl 1 ft. in diameter and full of water is being emptied through a hole in the bottom at the rate of 10 cu. in. per second. How fast is the surface of the water sinking when 100 cu. in. have run out? When the bowl is just half full?

5. If water flows from a hole in the bottom of a cylindrical can of radius r into another can of radius r', compare the vertical rates of rise and fall of the two water surfaces.

6. If a funnel is 8 in. wide and 6 in. deep and liquid flows from it at the rate of 5 cu. in. per minute, determine the time-rate of fall of the surface of the liquid.

7. Compare the vertical rates of the two liquid surfaces when water drains from a conical funnel into a cylindrical bottle. Compare the time-rate of flow from the funnel with the time-rate of the decrease of the wet perimeter.

8. If a wheel of radius R is turned by rolling contact with another wheel of radius R', compare their angular speeds and accelerations.

9. If a gear wheel moves a toothed rack so that a point of the rack moves according to the equation $s = 1 - t/2 + t^2/3$, what is the angular velocity and angular acceleration of the wheel at any time t, expressed in revolutions and seconds? Express the angular speed and the angular acceleration in terms of radians and seconds.

10. Compare the speed of a train with the speed of a point on the rim of a wheel; compare their accelerations.

11. If a point moves on a circle so that the arc described in time t is $s = t^2 - 1/t^2 + 1$, find the angular speed and acceleration of the radius drawn to the moving point.

12. A point moves along the parabola $y = 2x^2 - x$ in such a manner that the speed of the abscissa x is 1 ft./sec. Find the general expression for the speed of y; and find its value when $x = 2$; when $x = 4$.

13. In Ex. 12, find the horizontal and vertical accelerations, the total speed, the tangential acceleration, and the total acceleration. [See Exs. 16–18, p. 74.]

14. A point moves on the cubical parabola $y = x^3$ in such a way that the horizontal speed is 3 ft./sec. Express the vertical speed when $x = 6$. Find its value.

15. Find the quantities mentioned in Ex. 13, for the problem stated in Ex. 14.

16. If a person walks along a sidewalk at the rate of 3 mi. an hour toward the gate of a yard, how fast is he approaching a house in the yard which is 50 ft. from the gate in a line perpendicular to the walk, when he is 100 ft. from the gate? When 10 ft. from the gate?

17. Two ships start from the same point at the same time, one sailing due east at 10 knots an hour, the other due northwest at 12 knots an hour. How fast are they separating at any time? How fast, if the first ship starts an hour before the other?

18. If a ladder 8 ft. long rests against the side of a room, and its foot slips along the floor at a uniform rate of 1 ft./sec., how fast is the top descending when it is 6 ft. above the floor?

19. The sides of a right triangle about the right angle are originally 3 ft. and 5 ft. long, and grow at the rates of 3 in. and 2 in. a second, respectively. Express the lengths of these sides in terms of the time t, and calculate the rates of change per second of the area, and of the tangents of each of the acute angles of the triangle. What are these rates when $t = 1$ sec.? when $t = 10$ sec.? At what moment is the triangle isosceles?

20. If the radius of a sphere increases as the square root of the time; determine the time-rate of change of the surface and that of the volume; the acceleration of the surface and that of the volume.

21. Express the area between the x-axis and the line $y = x - 1$ from $x = 1$ to $x = x_0$ in terms of x_0. As x_0 changes show that the rate of change of this area is measured by $x_0 - 1$ or y_0.

22. If the space-time equation of a motion is $s = (a + bt)^{3/2}$, show that the speed varies inversely as the tangential acceleration.

23. What is the time-rate of change of the force acting on a body of mass m which moves on a straight line with the speed $v = at^2 + bt + c$?

24. If a projectile is fired at an angle of elevation α and with muzzle velocity v_0, its path (neglecting the resistance of the air) is the parabola

$$y = x \tan \alpha - \frac{gx^2}{2\, v_0^2 \cos^2 \alpha},$$

x being the horizontal distance and y the vertical distance from the point of discharge. Draw the graph, taking $g = 32$, $\alpha = 20°$, $v_0 = 2000$ ft./sec. Calculate dy in terms of dx. In what direction is the projectile moving when $x = 5000$ ft., 10,000 ft., 20,000 ft.? How high will it rise?

25. If in an experiment on compressing a gas it is known that pressure $\times$ volume = constant, and the time-rate of change of the pressure is $1 + t^2$, calculate the time-rate of change of the volume; compare the acceleration of the pressure and that of the volume.

26. If $p \cdot v = k$, compare dp/dt and dv/dt in general; compare d^2p/dt^2 and d^2v/dt^2.

27. If $p \cdot v^n = k$, compare dp/dt and dv/dt. [For air, in rapid compression, $n = 1.41$, nearly.]

28. If q is the quantity of one product formed in a certain chemical reaction in time t, it is known that $q = ck^2t/(1 + ckt)$. The time-rate of change of q is called the *speed* v of the reaction. Show that

$$v = \frac{ck^2}{(1 + ckt)^2} = c(k - q)^2.$$

Show also that the *acceleration* α of the reaction is

$$\alpha = -\frac{2\, c^2k^3}{(1 + ckt)^3} = -2\, c^2(k - q)^3.$$

CHAPTER V

REVERSAL OF RATES — INTEGRATION — SUMMATION

PART I. INTEGRALS BY REVERSAL OF RATES

54. Reversal of Rates. Up to this point, we have been engaged in finding rates of change of given functions. Often, the rate of change is known and the values of the quantity which changes are unknown; this leads to the problem of this chapter: *to find the amount of a quantity whose rate of change is known.*

Simple instances of this occur in every one's daily experience. Thus, if the rate r (in cubic feet per second) at which water is flowing into a tank is known, the total amount A (in cubic feet) of water in the tank at any time can be computed readily, — at least if the amount originally in the tank is known:

$$A = r \cdot t + C,$$

where t is the time (in seconds) the water has run, and C is the amount originally in the tank, *i.e.* C is the value of A at the time when $t = 0$.

If a train runs at 30 miles per hour, its total distance d, from a given point on the track, is

$$d = 30 \cdot t + C,$$

where t is the time (in hours) the train has run, and C is the original distance of the train from that point, *i.e.* C is the value of d when $t = 0$. (Notice that by regarding d as negative in one direction, this result is perfectly general; C may also be negative.)

If a man is saving \$ 100 a month, his total means is $100 \cdot n + C$, where n is the number of months counted, and C is his means at the beginning; *i.e.* C is his means when $n = 0$.

If the cost for operating a printing press is 0.01 ct. per sheet, the total expense of printing is

$$T = 0.01 \cdot n + C$$

where n is the number of copies printed, and where C is the first cost of the machine; *i.e.* C is the value of T when $n = 0$.

55. Principle Involved. Such simple examples require no new methods; they illustrate excellently the following fact:

The total amount of a variable quantity y at any stage is determined when its rate of increase and its original value C are known.*

We shall see that this remains true even when the rate itself is variable.

56. Illustrative Examples. The rate $R(x)$ at which any variable y increases with respect to an independent variable x is the *derivative* dy/dx; hence the general problem of § 54–55 may be stated as follows: *given the derivative dy/dx, to find y in terms of x.*

In many instances our familiarity with the rules for obtaining rates of increase (*differentiation*) enables us to set down at once a function which has a given rate of increase.

Example 1. Thus, in each of the examples given in § 54, the rate is constant; using the letters of this article:

$$\frac{dy}{dx} = R(x) = k,$$

where k is a known fixed number; it is obvious that a function which has this derivative is

$$(A) \qquad y = kx + C,$$

where C is any constant chosen at pleasure.

While the examples of § 54 can all be solved very easily without this new method, for those which follow it is at least very convenient. The value of C in any given example is found as in § 54; it represents the value of y when $x = 0$.

Example 2. Given $dy/dx = x^2$, to find y in terms of x.

Since we know that $d(x^3)/dx = 3x^2$, and since multiplying a function by a number multiplies its derivative by the same number, we should evidently take:

* This total amount is what is called in § 57 the **integral of the rate**; the word *integral* means precisely the "total" made from the rate, by its English derivation; compare the English words *entire, entirety, integrity, integer,* etc.

$$y = \frac{x^3}{3}, \quad \text{or else} \quad y = \frac{x^3}{3} + C\,; \quad \left[\text{check}: \; d\left(\frac{x^3}{3} + C\right) = x^2 dx\right],$$

where C is some constant. As in § 54, some additional information must be given to determine C. In a practical problem, such as Ex. 3, below, information of this kind is usually known.

Example 3. A body falls from a height 100 ft. above the earth's surface; given that the speed is $v = -gt$, find its distance from the earth in terms of the time t.

Let s denote the distance (in feet) of the body from the earth; we are given that

$$(1) \qquad v = \frac{ds}{dt} = -gt, \quad \text{or} \quad ds = v dt = -gt\,dt,$$

which is negative since s is decreasing. We know that $d(t^2) = 2\,t\,dt$; hence it is evident that we should take:

$$(2) \qquad s = -\frac{g}{2}t^2 + C; \quad [\text{check}: \; ds = -gt\,dt].$$

As the body starts to fall, $t = 0$ and $s = 100$; substituting these values in (2) we find $\quad 100 = 0 + C, \quad \text{or} \quad C = 100.$

In this problem, therefore, we have

$$s = -\frac{g}{2}t^2 + 100.$$

Example 4. Given $dy/dx = x^n$, to find y in terms of x.

Since we know that $d(x^{n+1}) = (n+1)x^n\,dx$, we should take

$$(B) \qquad y = \frac{1}{n+1}x^{n+1} + C; \quad [\text{check}: \; dy = x^n\,dx].$$

Since the rule for differentiation of a power was proved (§ 23, p. 38) for all positive and negative values of n, the formula (B) holds for all these values of n *except* $n = -1$; when $n = -1$ the formula (B) cannot be used because the denominator $n + 1$ becomes zero. (See § 78, p. 136.)

Special cases:

$$n = 1, \; \frac{dy}{dx} = x, \; y = \frac{1}{2}x^2 + C; \; \text{check}: \; d\left(\frac{1}{2}x^2\right) = x\,dx.$$

$$n = 0, \; \frac{dy}{dx} = 1, \; y = x + C; \; \text{check}: \; d(x) = 1 \cdot dx.$$

$$n = \frac{1}{2}, \; \frac{dy}{dx} = x^{1/2}, \; y = \frac{2}{3}x^{3/2} + C; \; \text{check}: \; d\left(\frac{2}{3}x^{3/2}\right) = x^{1/2}dx.$$

$$n = -2, \; \frac{dy}{dx} = x^{-2}, \; y = -x^{-1} + C; \; \text{check}: \; d(-1\,x^{-1}) = x^{-2}dx.$$

$$n = -\frac{1}{3}, \; \frac{dy}{dx} = x^{-1/3}, \; y = \frac{3}{2}x^{2/3} + C; \; \text{check}: \; d\left(\frac{3}{2}x^{2/3}\right) = x^{-1/3}dx.$$

Notice that these include $\sqrt{x}(=x^{1/2})$, $1/x^2(=x^{-2})$, etc.; other special cases are left to the student.

Example 5. Given $dy/dx = x^3 + 2x^2$, to find y in terms of x.

Since $d(x^4)/dx = 4x^3$ and $d(x^3)/dx = 3x^2$, and since the derivative of a sum of two functions is equal to the sum of their derivatives, it is evident that we should write *

$$y = \frac{x^4}{4} + \frac{2x^3}{3} + C.$$

The check is

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^4}{4} + \frac{2x^3}{3} + C\right) = x^3 + 2x^2;$$

such a check on the answer should be made in every exercise.

In general, as in this example, if the given rate of increase (derivative) is the sum of two parts, the answer is found by adding the answers which would arise from the parts taken separately, since the sum of the derivatives of two variables is always the derivative of their sum.

EXERCISES XIX. — REVERSAL OF RATES

1. Determine functions whose derivatives are given below; do not forget the additive constant; check each answer.

(*a*) $\dfrac{dy}{dx} = 4x.$ (*b*) $\dfrac{dy}{dx} = -5x.$ (*c*) $\dfrac{dy}{dx} = 3x^2.$ (*d*) $\dfrac{dy}{dx} = 2.$

(*e*) $\dfrac{dy}{dx} = -6x^2.$ (*f*) $\dfrac{dy}{dx} = -10x^5.$ (*g*) $\dfrac{dy}{dx} = -x^4.$ (*h*) $\dfrac{dy}{dx} = .01x^8.$

2. In the following exercises, remember that the derivative of a sum is the sum of the derivatives of the several terms; proceed as in Ex. 1.

(*a*) $\dfrac{dy}{dx} = 4 + 5x^2.$ (*b*) $\dfrac{dy}{dx} = 4x^2 - 2x + 3.$ (*c*) $\dfrac{ds}{dt} = t^3 - 4t + 7.$

(*d*) $\dfrac{dy}{dx} = 3x^5 - 8x^4.$ (*e*) $\dfrac{dy}{dx} = ax + b.$ (*f*) $\dfrac{ds}{dt} = at^2 + bt + c.$

(*g*) $\dfrac{dy}{dx} = .006x^2 - .004x^3 + .015x^4.$ (*h*) $\dfrac{ds}{dt} = -t^6 + 5t^4 - 6t^2 + 2.$

(*i*) $\dfrac{dy}{dx} = x^{-3}.$ (*j*) $\dfrac{ds}{dt} = t^2 + 1/t^2.$ (*k*) $\dfrac{dv}{dt} = 3t^{-2} + 4t^{-3}.$

(*l*) $\dfrac{dy}{dx} = x^{2/3}.$ (*m*) $\dfrac{dy}{dx} = 2x^{1/2} - 3x^{-1/2}.$ (*n*) $\dfrac{dp}{dv} = kv^{-2.41}.$

* In all the Examples of this paragraph, we have had an equation which involves dy/dx; such an equation is often called a **differential equation**, because it contains differentials. See also Chapter X.

3. As a train leaves a station, its speed v is proportional to the time; find the relation between the distance s passed over and the time.

[HINT. $v = ds/dt = kt$. Here and below, the unit of time is 1 sec.]

4. If in Ex. 3, $k=1/4$, find s when $t = 10$. What is the average speed during this time? Is the actual speed ever equal to this average speed? When? Try to make a rough estimate in advance.

5. Compare the speeds and the distances passed over by an express train which leaves a station with an increasing speed $v=t/2$, with that of a freight train which starts from a point 100 yd. ahead at the same instant with a speed $v = t/10$.

6. Determine v and s in terms of t for a bullet shot vertically upward with a speed 2000 ft./sec.

[HINT. Acceleration $= dv/dt = -g = -32.2$ ft./sec./sec.; $v = 2000$ when $t = 0$; $s = 0$ when $t = 0$. Neglect the air friction.]

7. How high will the bullet in Ex. 6 rise? How long will it remain in the air? Make a rough estimate in advance.

8. A car starts with a speed $v = t^2/12$; find s; how far will it go in 3 seconds?

9. A flywheel starts with an angular speed $\omega = .01\,t^2$ in radians per second. How long does it take to make the first revolution? How long for the next?

10. If a flywheel starts with a speed $\omega = .001\,t^3$, what is the time of the first revolution? of the second? of the tenth?

11. If the angular speed in radians per second of a wheel while stopping is $\omega = 100 - 10\,t$, how many revolutions will it make before it stops?

12. Determine the form of the surface of water in a rapidly rotated bucket from the fact that any vertical section through its lowest point has a slope $dy/dx = (\omega^2/g)x$, where x is measured horizontally and y vertically from the lowest point, ω is the angular speed in radians per second, and $g=32.2$. Plot the section when $\omega = 8$.

13. Determine a curve through $(0, 0)$ whose slope is proportional to x; to x^2; to $1 - x^3$.

14. Determine a curve through $(0, 0)$ and $(1, 2)$ whose flexion is proportional to x; to $1 + x^2$; one whose flexion is constant.

15. Determine the form of the upper surface of a beam if its flexion is constant, and if the beam rests on two fixed supports at a distance l from each other. See Ex. 20, p. 80.

57. Integral Notation. If the rate of increase $dy/dx = R(x)$ of one variable y with respect to another variable x *is given, a function* $y = I(x)$ *which has precisely this given rate of increase is called an* **indefinite integral** * **of the rate** $R(x)$, *and is represented by the symbol* †

(1) $$I(x) = \int R(x)\,dx;$$

that is,

(2) *if* $\dfrac{d}{dx}[I(x)] = R(x)$, *then* $I(x) = \int R(x)\,dx$,

or, what amounts to the same thing,

(3) *if* $d[I(x)] = R(x)\,dx$, *then* $I(x) = \int R(x)\,dx$.

The results of Examples 1, 2, 3, § 56, written with the new symbol, are, respectively,

[*A*] $$\int k\,dx = kx + C.$$

$$\int x^2 dx = x^3/3 + C.$$

$$s = \int v\,dt + C = \int -gt\,dt + C = -gt^2/2 + C.$$

The first equation of Example 3 holds in general:

[**I**] $$s = \int v\,dt + C, \text{ since } \frac{ds}{dt} = v.$$

The result obtained in (*B*), Example 4, § 56, gives

[*B*] $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1,$$

* The common English meaning of the word *integrate* is "to make whole again," "to restore to its entirety," "to give the sum or total." See any dictionary, and compare §§ 54–55.

To integrate a rate $R(x)$ is to find its *integral;* the process is called **integration**. Often the rate function $R(x)$ which is integrated is called the **integrand**; thus the first part of equation (2) may be read: "*the derivative of the integral is the integrand.*" This is the property used in checking answers.

The first equation in (2) and the first in (3) are *differential* equations. See footnote, p. 94.

† Note that dx is *part of the symbol.* As a blank symbol, it is $\int$ (blank) dx; the function $R(x)$ to be integrated (*i.e.* the integrand) is inserted in place of the blank. The origin of this symbol is explained in § 67, p. 117.

for all positive and negative integral and fractional values of n *except* $n = -1$, for which see § 78, p. 136.

As examples of the many special cases, we write:

$$n = 1, \quad \int x\,dx \quad = \frac{x^2}{2} + C.$$

$$n = 0, \quad \int x^0 dx \quad = \int 1\,dx \quad = \int dx \qquad = x + C.$$

$$n = \frac{1}{2}, \quad \int x^{1/2} dx \quad = \int \sqrt{x}\,dx = \tfrac{2}{3} x^{3/2} + C = \tfrac{2}{3}\sqrt{x^3} + C.$$

$$n = -2, \quad \int x^{-2} dx \quad = \int \frac{1}{x^2}\,dx = -x^{-1} + C = -\frac{1}{x} + C.$$

$$n = -\frac{1}{3}, \quad \int x^{-1/3} dx = \int \frac{1}{\sqrt[3]{x}}\,dx = \tfrac{3}{2} x^{2/3} + C = \tfrac{3}{2}\sqrt[3]{x^2} + C.$$

From Example 5:

$$\int (x^3 + 2\,x^2)dx = \int x^3 dx + \int 2\,x^2 dx \qquad = \frac{x^4}{4} + \frac{2\,x^3}{3} + C.$$

The general principle used in this example is that the *integral of a sum of two functions is the sum of their integrals:*

$$[C] \qquad \int [R(x) + S(x)]\,dx = \int R(x)\,dx + \int S(x)\,dx,$$

which is true because the derivative of the sum $R(x) + S(x)$ is the sum of the derivatives:

$$d[R + S]/dx = dR/dx + dS/dx.$$

The rules (A), (B), (C) are sufficient to integrate a large number of functions, including certainly all *polynomials* in x.

EXERCISES XX. — NOTATION — INDEFINITE INTEGRALS

1. Express the value of y if $dy/dx = 4\,x^2 + 3\,x$ by means of the new sign $\int$ (——)dx. Also find y. Check.

2. If dy/dx has any one of the following values, express y, first by use of the new sign $\int$ (——)dx and then directly in terms of x. Check the final answers. Do not omit the arbitrary constant.

(*a*) x^2. (*c*) x^4. (*e*) $x^2 - 2$. (*g*) $x^2 - 2\,x + 3$. (*i*) $ax + b$.
(*b*) x^3. (*d*) $2\,x + 3$. (*f*) 7. (*h*) $x^2 - x^4$. (*j*) $\sqrt{x}$.

3. In many examples it is profitable first to expand the given expression in a sum of powers; proceed as in Ex. 2, and find y if dy/dx has any of the following values:

(a) $x(1+x)$. (e) $(1+x^2)(1-x^2)$. (i) $(1+x)(1-x^{-3})$.
(b) $(x^3+4x^2)\div x$. (f) $(2-3x)(4+x)$. (j) $x^{2/3}(x+x^2)$.
(c) $4(x+2)^2$. (g) $x^{1/2}(1+x)$. (k) $(x^3+2x^2+x)^{1/2}$.
(d) $2x^2(3-4x^2)$. (h) $(1-4x)\sqrt{x}$. (l) $(x^2+2)^2(2x^{1/2}+3)$.

4. Integrate the following expressions:

(a) $\int x^3dx$. (d) $\int(1+v)dv$. (g) $\int x^{-2}dx$.
(b) $\int 3t^4dt$. (e) $\int(3x^2-4x^3)dx$. (h) $\int 3t^{-4}dt$.
(c) $\int 7s^{-5}ds$. (f) $\int(10-9x^{-3})dx$. (i) $\int t^{1/5}dt$.
(j) $\int(3u+5u^2+7u^3)du$. (k) $\int(7/x^{12}+8x^{-10}-10/x^4)dx$.
(l) $\int 2\sqrt[3]{x^2}\,dx$. (m) $\int(5s^4-3s^2+2)\,ds$. (n) $\int(t^{8/3}+10t^{5/2})\,dt$.

5. Integrate the following expressions, making use of the principle of Ex. 3:

(a) $\int(1-t)^2\,dt$. (g) $\int\sqrt{x}(a+bx)\,dx$.
(b) $\int x(1+\sqrt{x})\,dx$. (h) $\int x^n(a+bx)\,dx$.
(c) $\int s(1-\sqrt{s})^2\,ds$. (i) $\int(a+bx)^2\,dx$.
(d) $\int t^2(1-t^2)\,dt$. (j) $\int(t^{2.5}-2)\,t^{-5}\,dt$.
(e) $\int x^{-4}(1+x+x^2)\,dx$. (k) $\int x^{1.4}(1+x^2)^2\,dx$.
(f) $\int x(a+bx)\,dx$. (l) $\int\sqrt{t}(1+2t^2)^2\,dt$.

6. Powers of linear expressions may be treated without expanding. Find a function whose derivative is $(x+1)^2$ by analogy with the function whose derivative is x^2.

7. Can y be found when $dy/dx=(x+3)^2$ by the same analogy? Can y be found when $dy/dx=(2x+3)^2$? Be sure to *check* your answers. If they are wrong, put in the proper factor to correct the error.

8. Find y when dy/dx has one of the following values:

(a) $(3x-2)^3$. (b) $(2x+1)^{1/2}$. (c) $(5x-4)^{-3}$.

58. Fundamental Theorem. We have seen that such functions as x^2, $x^2 + 5$, $x^2 + C$, where C is any constant, have the same derivative $2x$. If the rate of increase (derivative) of y with respect to x is given, there may be several answers for y in terms of x; thus if $dy/dx = 2x$, the answers $y = x^2$, $y = x^2 + 5$, $y = x^2 + C$ are all correct solutions; to decide which one is wanted, additional information is needed, as in § 54 and § 56 (Exs. 2, 3, etc.).

However, except for the additive constant C, all answers coincide; for practical purposes, there is but *one* answer. Stated precisely, this is the **Fundamental Theorem of Integral Calculus**:

If the rate of increase

$$\frac{dy}{dx} = R(x) \tag{1}$$

of a variable quantity y which depends on x is given, then y is determined as a function of x, $I(x)$, except for a constant term:

$$y = \int R(x)dx + C = I(x) + C. \tag{2}$$

Stated in different words this theorem is:

The difference between any two functions $I(x)$ and $J(x)$ whose derivatives are equal, is a constant.

Let this difference be

$$D(x) = I(x) - J(x);$$

then

$$\frac{dD(x)}{dx} = \frac{dI(x)}{dx} - \frac{dJ(x)}{dx} = 0.$$

Since the rate of increase (derivative) of $D(x)$ is *zero*, $D(x)$ neither increases nor decreases for any value of x; hence $D(x)$ is a *constant*, as was to be proved.*

The constant C which occurs in the answers is always to be determined by additional information, as in § 54 and § 56.

* Graphically, the "curve" which represents $D(x)$ has its tangent horizontal at every point, — such a "curve" is necessarily a horizontal straight line: $D(x) =$ constant. (See also § 131.)

59. Definite Integrals. In applications, we often care little about the actual total; it is rather the difference between two values which is important.

Thus, in a motion, we care little about the real total distance a body has traveled since the creation of the universe; it is rather the distance it has traveled *between two given instants.*

If a body falls from any height, the distance it falls is (Ex. 3, p. 93)

$$s = \int v\,dt + C = \int gt\,dt + C = \frac{gt^2}{2} + C,$$

where s is counted downwards.

The value of s when $t = 0$ is $s]_{t=0} = C$; the value of s when $t = 1$ is $s]_{t=1} = g/2 + C$. The distance traversed *in the first second* is found by subtracting these values:

$$s\Big]_{t=0}^{t=1} = s\Big]_{t=1} - s\Big]_{t=0} = \left(\frac{g}{2} + C\right) - C = \frac{g}{2} = 16.1 \text{ ft.},$$

where $s\Big]_{t=0}^{t=1}$ means the space passed over between the times $t = 0$ and $t = 1$.

In this calculation, we care little about where s is counted from; or its *total* value. The result is the same for all bodies dropped from any height.

Likewise, the space passed over between the times $t = 2$ and $t = 5$ is

$$s\Big]_{t=2}^{t=5} = s\Big]_{t=5} - s\Big]_{t=2} = \left(\frac{g \cdot 25}{2} + C\right) - \left(\frac{g \cdot 4}{2} + C\right)$$

$$= \frac{g \cdot 25}{2} - \frac{g \cdot 4}{2} = g \cdot \frac{21}{2} = 338 \text{ (ft.)}.$$

In general the distance traversed between the times $t = a$ and $t = b$ is

$$s\Big]_{t=a}^{t=b} = s\Big]_{t=b} - s\Big]_{t=a} = \left(g\frac{b^2}{2} + C\right) - \left(g\frac{a^2}{2} + C\right) = g\frac{b^2}{2} - g\frac{a^2}{2} = \frac{g}{2}(b^2 - a^2).$$

The advantage realized in this example in eliminating C can be gained in all problems:

The numerical value of the total change in a quantity between two values of x, $x = a$ and $x = b$, can be found if the rate of change $dy/dx = R(x)$ is given. For, if

$$y = I(x) = \int R(x)\,dx + C,$$

the value of y for $x = a$ is

$$y\Big]_{x=a} = I(a) = \Big[\int R(x)\,dx\Big]_{x=a} + C,$$

and the value of y for $x = b$ is

$$y\Big]_{x=b} = I(b) = \Big[\int R(x)\,dx\Big]_{x=b} + C.$$

The total change in y between the values $x = a$ and $x = b$ is

$$\begin{aligned} y\Big]_{x=a}^{x=b} = y\Big]_{x=b} - y\Big]_{x=a} &= I(b) - I(a) \\ &= \Big[\int R(x)dx\Big]_{x=b} - \Big[\int R(x)dx\Big]_{x=a}. \end{aligned}$$

This difference, found by subtracting the values of the indefinite integral at $x = a$ from its value at $x = b$, is called the **definite integral** *of $R(x)$ between $x = a$ and $x = b$; and is denoted by the symbol:*

$$\begin{aligned} \int_{x=a}^{x=b} R(x)\,dx &= \Big[\int R(x)\,dx\Big]_{x=b} - \Big[\int R(x)\,dx\Big]_{x=a} \\ &= I(b) - I(a). \end{aligned}$$

It should be noticed that, in subtracting, the unknown constant C has disappeared completely; this is the reason for calling this form *definite*.

Example 1. Given $dy/dx = x^3$, find the total change in y from $x = 1$ to $x = 3$.

Since

$$y = \int x^3\,dx = x^4/4 + C,$$

it follows that

$$y\Big]_{x=1}^{x=3} = y\Big]_{x=3} - y\Big]_{x=1} = \frac{x^4}{4}\Big]_{x=3} - \frac{x^4}{4}\Big]_{x=1} = 20.$$

Interpreted as a problem in motion, where x means time and y means distance, this would mean: the total distance traveled by a body between the end of the first second and the end of the third second, if its speed is the cube of the time, is twenty units.

Interpreted graphically, a curve whose slope m is given by the equation $m = x^3$, rises 20 units between $x = 1$ and $x = 3$. The equation of the curve is $y = x^4/4 + C$.

EXERCISES XXI. — DEFINITE INTEGRALS

1. If water pours into a tank at the rate of 200 gal. per minute, how much enters in the first ten minutes? how much from the beginning of the fifth minute to the beginning of the tenth minute?

2. If a train is moving at a speed of 20 mi. per hour, how far does it go in two hours? Does this necessarily mean the distance from its last stop?

3. If a train leaves a station with a variable speed $v = t/4$ (ft./sec.), find s in terms of t. How far does the train go in the first ten seconds? How far from the beginning of the fifth to the beginning of the tenth second?

4. A falling body has a speed $v = gt$, where t is measured from the instant it falls. How far does it go in the first five seconds? How far between the times $t = 3$ and $t = 7$?

5. A wheel rotates with a variable speed (radians/sec.) $\omega = t^2/100$. How many revolutions does it make in the first fifteen seconds? How many between the times $t = 5$ and $t = 20$?

6. From the following rates of change determine the total change in the functions between the limits indicated for the independent variable. Interpret each result geometrically and as a problem in motion, and write your work in the notation used in the text:

(a) $\dfrac{dy}{dx} = x$, $x = 1$ to $x = 2$.

(b) $\dfrac{dy}{dx} = \dfrac{1}{4}x^2$, $x = -2$ to $x = 2$.

(c) $\dfrac{dy}{dx} = \dfrac{x^3}{12}$, $x = -2$ to $x = 2$.

(d) $\dfrac{dy}{dx} = 1 - x^2$, $x = 0$ to $x = 10$.

(e) $\dfrac{ds}{dt} = \dfrac{1}{t^2}$, $t = 10$ to $t = 100$.

(f) $\dfrac{ds}{dt} = \dfrac{t^4 - 2}{t^2}$, $t = 1$ to $t = 3$.

(g) $\dfrac{dv}{dt} = \dfrac{(1+t)^2}{t^{3/2}}$, $t = 16$ to $t = 25$.

(h) $\dfrac{dv}{dt} = \dfrac{2t^{-1} - 3t^{-2}}{t^2}$, $t = 0.1$ to 0.01.

(i) $\dfrac{d\theta}{dt} = \sqrt{t\sqrt{t}}$; $t = 0$ to $t = .01$.

(j) $\dfrac{d\theta}{dt} = \dfrac{a^2}{t^2} - \dfrac{a^3}{t^3}$, $t = a$ to $t = 2a$.

7. Determine the values of the following definite integrals. [In cases where no misunderstanding could possibly arise, only the numerical values of the limits are given. In every such case, the numbers stated as limits are values of the variable whose differential appears in the integral.]

(a) $\int_{x=0}^{x=2} 2\,x\,dx.$ (f) $\int_0^8 x^{2/3}\,dx.$ (k) $\int_1^{-8} s^{2/3}(s^2-2\,s)\,ds.$

(b) $\int_{x=-2}^{x=+2} 2\,x\,dx.$ (g) $\int_{t=3}^{t=5}(1+t)\,dt.$ (l) $\int_{\theta=0}^{\theta=.01}\frac{d\theta}{100}.$

(c) $\int_{x=1}^{x=2} 2\,x^2\,dx.$ (h) $\int_{t=-2}^{t=0} 2(t^2-1)\,dt$ (m) $\int_{\theta=10}^{\theta=100}(.01+.02\,\theta)\,d\theta.$

(d) $\int_{x=-a}^{x=a} 2\,x^2\,dx.$ (i) $\int_{-2}^{3}(1+s+s^2)\,ds.$ (n) $\int_8^{27}\left(\sqrt[3]{\theta}+\frac{1}{\sqrt[3]{\theta}}\right)d\theta.$

(e) $\int_0^4\sqrt{x}\,dx.$ (j) $\int_1^3\frac{1+s^4}{s^2}\,ds.$ (o) $\int_{2.3}^{7.4} v^{-1.41}\,dv.$

8. A stone falls with a speed $v = gt + 10$. Find s in terms of t and find the distance passed over between the times $t = 2$ and $t = 7$.

9. A bullet is fired vertically with a speed $v = -gt + 1500$. How far does it go in ten seconds? How high does it rise? How long is it in the air? Make rough estimates of the answers in advance.

10. For any falling body, j = acceleration = g = const. Find the increase in speed in ten seconds. Does it matter what particular ten seconds are chosen?

11. If, in Ex. 10, the speed is 100 ft./sec. when $t = 5$, what is the speed when $t = 15$? When will the speed be 250 ft./sec.? Express v in terms of t.

60. Area under a Curve. Consider the area A under any curve $y = f(x)$, between the x-axis and the curve, and between a fixed vertical line through a fixed point F, $(x = k)$, and a movable vertical line through a movable point M, $(x = x)$.

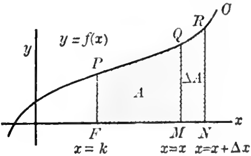


FIG. 22.

As M moves to N, x increases by an amount $\Delta x = MN$, and A increases by an amount ΔA = area of $MNRQ$. The average rate of increase of A is $\Delta A \div \Delta x$, which is equal to some height between the extremes of the values of y along MN. As Δx approaches zero, this intermediate height approaches the height at M; *and the instantaneous rate of increase in A is*

(1) $$\frac{dA}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta A}{\Delta x} = \lim_{MN \doteq 0} \frac{MNRQ}{MN} = MQ = y = f(x);$$

the rate of increase of A with respect to x is the height of the curve.

It follows that

(2) $$A = \int y\,dx + C = \int f(x)\,dx + C;$$

and the area A between any two fixed values of x, $x = a$ and $x = b$ is the definite integral:

(3) $$A\Big]_{x=a}^{x=b} = A\Big]_{x=b} - A\Big]_{x=a} = \int_{x=a}^{x=b} f(x)\,dx.$$

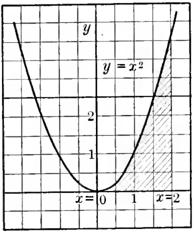


FIG. 23.

Example 1. To find the area under the curve* $y = x^2$ between the points where $x = 0$ and $x = 2$.

We have, by (2)

$$A = \int y\,dx + C = \int x^2\,dx + C = \frac{x^3}{3} + C,$$

where A is counted from any fixed back boundary $x = k$ we please to assume, up to a movable boundary $x = x$.

The area between $x = 0$ and $x = 2$ is given by subtracting the value of A for $x = 0$ from the value of A for $x = 2$:

$$A\Big]_{x=0}^{x=2} = A\Big]_{x=2} - A\Big]_{x=0} = \int_{x=0}^{x=2} x^2\,dx = \frac{x^3}{3}\Big]_{x=2} - \frac{x^3}{3}\Big]_{x=0} = \frac{8}{3}.$$

Likewise the area under the curve between $x = 1$ and $x = 3$ is

$$A\Big]_{x=1}^{x=3} = \int_{x=1}^{x=3} x^2\,dx = \frac{x^3}{3}\Big]_{x=3} - \frac{x^3}{3}\Big]_{x=1} = 8\tfrac{2}{3};$$

and the area under the curve between any two vertical lines $x = a$ and $x = b$ is

$$A\Big]_{x=a}^{x=b} = \int_{x=a}^{x=b} x^2\,dx = \frac{x^3}{3}\Big]_{x=b} - \frac{x^3}{3}\Big]_{x=a} = \frac{b^3 - a^3}{3}.$$

* The phrase "the area under the curve" is understood in the sense used in the first sentence of § 60. When the curve is below the x-axis, this area is counted as negative.

EXERCISES XXII. — AREAS

[Draw a figure and estimate the answer in advance, whenever possible.]

1. Find the area under each of the following curves between the ordinates $x = 0$ and $x = 1$; between $x = 2$ and $x = 5$:

(*a*) $y = 3x^2$. (*d*) $y = \sqrt{x}$. (*g*) $x^2y = 1$. (See § 111.)
(*b*) $y = x^3$. (*e*) $y = 1/\sqrt{x}$. (See § 111.) (*h*) $y = x^2 + 3x - 4$.
(*c*) $y = x^4/10$. (*f*) $y = (1 - x^2)$. (*i*) $y = x(1 - x)^2$.

2. Find the area between the line $y = 2x$ and the parabola $y = x^2$.

3. Find the area between $y = x$ and $y = \sqrt{x}$.

4. Show that $y = x^2$ and $y^2 = x$ trisect the unit square whose diagonal joins the points (0, 0) and (1, 1).

5. Find the area between $y = x^2$ and $y = x^3$; and show that it is the same as that under the curve $y = x^2 - x^3$.

6. Find the areas under each of the following curves:

(*a*) $y = x^3 + 6x^2 + 15x$, ($x = 0$ to 2; $x = -2$ to $+2$; $x = -a$ to $+a$).
(*b*) $y = x^{2/3}$, ($x = 0$ to 8; $x = -1$ to $+1$; $x = -a$ to $+a$).
(*c*) $y = x^2 + 1/x^2$, ($x = 1$ to 3; $x = 2$ to 5; $x = a$ to b).

7. Find the area A under the line $y = 2x + 3$ between $x = 0$ and $x = x$ (any value) by geometry; show directly that $dA/dx = y$.

8. Find geometrically the area A under the line $x + y + 2 = 0$ between $x = 0$ and $x = x$; show directly that $dA/dx = y$.

9. Show that the area $A*$ bounded by a curve $x = \phi(y)$ the y-axis, and the two lines $y = a$ and $y = b$ is

$$A* = \int_{y=a}^{y=b} \phi(y)dy.$$

10. Calculate the area between the y-axis, the curve $x = y^2$, and the lines $y = 0$ and $y = 1$. Compare this answer with that of Ex. 3.

11. Find the area between the curve $y = x^3$ and each of the axes separately, from the origin to a point (k, k^3). Show that their sum is k^4.

12. Find the area in the first quadrant between $y = x^2 + 3x$, the y-axis, and the lines $y = 0$, $y = 4$, by subtracting from a certain rectangle the area between $y = x^2 + 3x$, the x-axis, and the lines $x = 0$, $x = 1$.

13. Find the area in the first quadrant between the curve $y = x^2 + 2x - 7$, the y-axis, and the lines $y = 2$, $y = 9$, by the method of Ex. 12.

61. Lengths of Curves. Let s represent the length of the arc FM of a curve C whose equation is $y=f(x)$, between a fixed point F and a moving point M, on the curve.

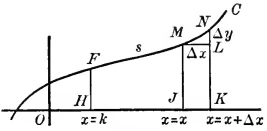


FIG. 24.

As the point M moves on to N, the value of x increases by an amount $\Delta x = JK = ML$, y by an amount $\Delta y = LN$, and s by an amount $\Delta s =$ arc MN.

The chord MN is given by the Pythagorean theorem:

(1) $$[\text{chord } MN]^2 = \overline{ML}^2 + \overline{LN}^2 = \overline{\Delta x}^2 + \overline{\Delta y}^2.$$

The instantaneous rate of increase of the arc s with respect to x is

(2) $$\frac{ds}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta s}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{\text{arc } MN}{\Delta x};$$

this limit can be found by the fundamental fact (§ 12) that the limit of the ratio of an arc to its subtended chord is unity.*

(3) $$\frac{ds}{dx} = \lim_{\Delta x \doteq 0} \frac{\text{arc } MN}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{\text{chord } MN}{\Delta x},$$

since $$\lim_{MN \doteq 0} \frac{\text{arc } MN}{\text{chord } MN} = 1;$$

hence

(4) $$\left[\frac{ds}{dx}\right]^2 = \left[\lim_{\Delta x \doteq 0} \frac{(\text{chord } MN)^2}{\overline{\Delta x}^2}\right] = \lim_{\Delta x \doteq 0} \frac{\overline{\Delta x}^2 + \overline{\Delta y}^2}{\overline{\Delta x}^2}$$

$$= \lim_{\Delta x \doteq 0}\left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right] = 1 + \left(\frac{dy}{dx}\right)^2$$

or

(5) $$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left[\frac{df(x)}{dx}\right]^2} = \sqrt{1+m^2}.$$

It follows that the total arc is

(6) $$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx + C = \int \sqrt{1+m^2}\, dx + C,$$

* This fact is the pith of the argument; it contains the essence of the definition of what we mean by the length of an arc of a curve. See § 12.

and that the length of the arc between any two points at which $x = a$ and $x = b$, respectively, is

$$(7)\qquad s\Big]_{x=a}^{x=b} = \int_{x=a}^{x=b} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{x=a}^{x=b} \sqrt{1+m^2}\,dx.$$

62. Motion on a Curve. Parameter Forms. The equation (5) of § 61 is often written in the form

$$(8)\qquad ds^2 = dx^2 + dy^2,$$

which is readily remembered through the suggestiveness of the triangles MLN and MLP of Fig. 25, in which $ds = MP$. Equation (8) will be called the **Pythagorean differential formula**. Since $m = \tan\alpha$ (Fig. 12, p. 50), the quantity $\sqrt{1+m^2}$ is equal to $\sec\alpha$. In particular, $ds/dx = \sec\alpha$, whence $dx/ds = \cos\alpha$. Likewise, $dy/ds = \sin\alpha$.

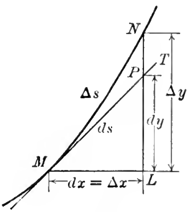


FIG. 25.

If a point M moves along a curve, its total speed v is ds/dt, its horizontal speed v_x is dx/dt; its vertical speed v_y is dy/dt. Dividing both sides of equation (8) by $(dt)^2$, we find

$$(9)\qquad \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2, \text{ or } v^2 = v_x{}^2 + v_y{}^2:$$

the square of the total speed *is the sum of the squares of the horizontal and the vertical speeds;* and $v_x = v\cos\alpha$, $v_y = v\sin\alpha$.

The equation (8) may be used in case the equations of the curve are given in parameter form

$$(10)\qquad x = f(t), \qquad y = \phi(t),$$

whether the parameter t represents the time or some other convenient quantity. The length of an arc of a curve whose equations are given in the form (10) is

$$(11)\qquad s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt + C.$$

63. Illustrative Examples.

Example 1. A point moves along a curve $y = x^2$. Express ds/dx and write the integral which represents the length of the curve.

By the Pythagorean formula $ds^2 = dx^2 + dy^2$; but $dy = 2\,x\,dx$; hence

$$ds^2 = (1 + 4\,x^2)\,dx^2, \quad \text{or} \quad \frac{ds}{dx} = \sqrt{1 + 4\,x^2}.$$

The length s of any arc between points where $x = a$ and $x = b$ is

$$s\Big]_{x=a}^{x=b} = \int_{x=a}^{x=b} \sqrt{1 + 4\,x^2}\,dx\,;$$

but since we have never had a function whose derivative is $\sqrt{1 + 4\,x^2}$, this integral cannot now be found. [See, however, Ex. 16, p. 129, and § 106.]

The speeds v, v_x, and v_y are given by the relations:

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt} = 2\,x\frac{dx}{dt} = 2\,xv_x, \quad v = \sqrt{v_x{}^2 + v_y{}^2} = \sqrt{1 + 4\,x^2}\,v_x.$$

If v_x is given, the other two can be found; thus, if v_x is a constant k,

$$v_x = k, \quad v_y = 2\,kx, \quad v = k\sqrt{1 + 4\,x^2}.$$

Example 2. Find ds, v, s for the curve $y^2 = x^3$ and find the length of the arc from the origin to the point where $x = 5$.

Since $y^2 = x^3$, we have $2\,y\,dy = 3\,x^2\,dx$, or $4\,y^2\,dy^2 = 9\,x^4\,dx^2$, or $dy^2 = \frac{9}{4}\,x\,dx^2$; and $ds^2 = dx^2 + dy^2 = (1 + \frac{9}{4}\,x)\,dx^2$. It follows that the speed of a moving point is

$$v^2 = \left(\frac{ds}{dt}\right)^2 = v_x{}^2 + v_y{}^2 = \left(1 + \frac{9}{4}\,x\right)v_x{}^2\,;$$

and that the length of the arc is

$$s = \int \sqrt{1 + \tfrac{9}{4}\,x}\,dx + C = \tfrac{8}{27}(1 + \tfrac{9}{4}\,x)^{3/2} + C,$$

since

$$\frac{d}{dx}\left[\frac{8}{27}\left(1 + \frac{9}{4}\,x\right)^{3/2}\right] = \sqrt{1 + \frac{9}{4}\,x}\,;$$

hence the length between the origin and the point where $x = 5$ is

$$s\Big]_{x=0}^{x=5} = \int_{x=0}^{x=5} \sqrt{1 + \tfrac{9}{4}\,x}\,dx = \tfrac{8}{27}\left(1 + \tfrac{9}{4}\,x\right)^{3/2}\Big]_{x=0}^{x=5} = \tfrac{335}{27}.$$

Example 3. Find ds, v, s for the curve represented by the equations

$$x = t^2 + 5, \quad y = \tfrac{1}{6}(4\,t + 1)^{3/2}.$$

We find $ds^2 = dx^2 + dy^2 = (2\,t\,dt)^2 + [(4\,t + 1)^{1/2}\,dt]^2 = (2\,t + 1)^2\,dt^2$; whence $ds = (2\,t + 1)\,dt$, or $v = ds/dt = 2\,t + 1$, and

$$s = \int \frac{ds}{dt}\,dt + C = \int (2\,t + 1)\,dt + C = t^2 + t + C.$$

Between the points where $t = 0$ and where $t = 2$ [*i.e.* the points ($x = 5$; $y = 1/6$) and ($x = 9$, $y = 27/6$)], the length of the arc is

$$s\Big]_{t=0}^{t=2} = \int_{t=0}^{t=2}(2t+1)dt = \Big[t^2 + t + C\Big]_{t=0}^{t=2} = (4+2+C) - C = 6.$$

Our present ability to recognize derivatives enables us to integrate comparatively few of the square root forms that occur in these length integrals. We shall be able to deal with these forms more readily in Chapter VI.

EXERCISES XXIII. — LENGTH — TOTAL SPEED

1. Determine by integration the lengths of the following curves, each between the limits $x = 1$ to $x = 2$, $x = 2$ to $x = 4$, $x = a$ to $x = b$. Check the first three geometrically:

(*a*) $y = 2x - 1$. (*c*) $y = mx + c$. (*e*) $y = \frac{1}{3}(2x - 1)^{3/2}$.
(*b*) $y = 3 + 4x$. (*d*) $y = \frac{2}{3}(x - 1)^{3/2}$. (*f*) $y = \frac{1}{6}(4x - 1)^{3/2}$.

2. Find ds, and the length s of the path of each of the following motions, between the given limits. Find the speed v at each end of the arc.

(*a*) $x = 1 + t$, $y = 1 - t$; $t = 0$ to $t = 2$.
(*b*) $x = (1 + t)^{3/2}$, $y = (1 - t)^{3/2}$; $t = 0$ to $t = 1$.
(*c*) $x = (1 - t)^2$, $y = 8t^{3/2}/3$; $t = 0$ to $t = 9$.
(*d*) $x = 1 + t^2$, $y = t - t^3/3$; $t = 0$ to $t = 5$.
(*e*) $x = 2/t$, $y = t + 1/(3t^3)$; $t = a$ to $t = b$.

3. If a point moves on the circle $x^2 + y^2 = 1$, show that $x(dx/dt) + y(dy/dt) = 0$, and that $v^2 = [dx/dt]^2/y^2 = [dy/dt]^2/x^2$.

4. If a point moves on the circle $x^2 + y^2 = 1$ *with constant speed* $v = k$, show that $dx/dt = \pm ky$ and $dy/dt = \pm kx$, where the sign $\pm$ depends on the sense of the motion.

5. If a point moves on the hyperbola $xy = 1$, show that the horizontal and the vertical speeds v_x and v_y are connected by the relation $xv_y + yv_x = 0$; and that $v^2 = v_x^2(x^2 + y^2)/x^2 = v_y^2(x^2 + y^2)/y^2$.

6. If a point moves on the curve $y^2 = x$, show that $v^2 = (1 + 4y^2)v_y^2$.

7. Determine the path described when the x and y speeds are as below, if the point is at (0, 0) when $t = 0$. Find the length of the arc traversed from $t = 0$ to $t = 9$. What is the speed at each end of these arcs?

(*a*) $\begin{cases} \dfrac{dx}{dt} = \sqrt{t}; \\ \dfrac{dy}{dt} = 1. \end{cases}$ (*b*) $\begin{cases} \dfrac{dx}{dt} = 2\sqrt{t}; \\ \dfrac{dy}{dt} = 1 - t. \end{cases}$ (*c*) $\begin{cases} \dfrac{dx}{dt} = \sqrt{1 + t}; \\ \dfrac{dy}{dt} = \sqrt{1 - t}. \end{cases}$

PART II. INTEGRALS AS LIMITS OF SUMS

64. Step-by-step Process. The total amount of a variable quantity whose rate of change (derivative) is given [*i.e.* the integral of the rate] can be obtained in another way.

For example, imagine a train whose speed is increasing. The distance it travels cannot be found by multiplying the speed by the time; but we can get the total distance *approximately* by steps, computing (approximately) the distance traveled in each second as if the train were actually going at a constant speed during that second, and adding all these results to form a total distance traveled.

If the speed increases steadily from zero to 30 mi. per hour, in 44 sec., that is, from zero to 44 ft. per second in 44 sec., the increase in speed each second (acceleration) is 1 ft. per second. Hence the speeds at the beginnings of each of the seconds are 0, 1, 2, 3, ⋯ , etc.

Using the speeds as approximately correct during one second each, we should find the total distance (approximately)

$$s = 0 + 1 + 2 + 3 + \cdots + 42 + 43 = \frac{43 \cdot 44}{2} = 946,$$

which is evidently a little too low.

If we used as the speed during each second the speed at the end of that second, we should get (approximately)

$$s = 1 + 2 + 3 + 4 + \cdots + 43 + 44 = \frac{44 \cdot 45}{2} = 990,$$

which is evidently too high. But these values differ only by 44 ft.; and we are sure that the desired distance is between 946 and 990 ft.

If we reduce the length of the intervals, the result will be still more accurate; thus if, in the preceding example, the distances be computed by half seconds, it is easily shown that the distance is between 957 ft. and 979 ft.; if the steps are taken 1/10 second each, the distance is found to be between 965.8 ft. and 970.2 ft.

Evidently, the exact distance is the *limit* approached by this step-by-step summation as the steps Δt approach zero:

$$s\Big]_{t=0}^{t=44} = \int_{t=0}^{t=44} v\,dt = \int_{t=0}^{t=44} t\,dt = \frac{t^2}{2}\Big]_{t=0}^{t=44} = 968.$$

We note particularly that the two results for s are surely equal; hence we obtain the important result:

$$\int_{t=0}^{t=44} v\,dt = \lim_{\Delta t \doteq 0} \left\{ v\Big]_{t=0} \cdot \Delta t + v\Big]_{t=\Delta t} \cdot \Delta t + v\Big]_{t=2\Delta t} \cdot \Delta t + \cdots \right\}.$$

65. Approximate Summation. This step-by-step process of summation to find a given total is of such general application, and is so valuable even in cases where no limit is taken, that we shall stop to consider a few examples, in which the methods employed are either obvious or are indicated in the discussion of the example.

Thus, areas are often computed approximately by dividing them into convenient strips. We have seen, § 60, that if A denotes the area under a curve between $x = a$ and $x = b$, then the rate of increase of A is the height h of the curve:

$$\frac{dA}{dx} = h = R(x),$$

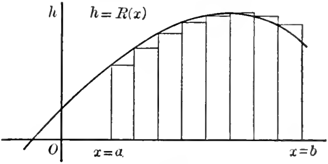


FIG. 26.

where $R(x)$ is the rate of increase of A, and is also the height of the curve.

For a parabola, $h = x^2$, we may find the area A approximately between $x = -1$ and $x = 2$ by dividing that interval into smaller pieces and computing (approximately) the areas which stand on those pieces as if the height h were constant throughout each piece. If, for example, we divide the area A into six strips of equal width, each 1/2 unit wide, and if we take the height throughout each one to be the height at the left-hand corner, the total area is (approximately)

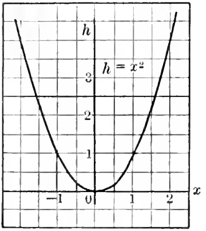


FIG. 27.

$$(-1)^2 \cdot \tfrac{1}{2} + (-\tfrac{1}{2})^2 \cdot \tfrac{1}{2} + 0 \cdot \tfrac{1}{2} + (\tfrac{1}{2})^2 \tfrac{1}{2} + (+1)^2 \tfrac{1}{2} + (\tfrac{3}{2})^2 \tfrac{1}{2} = 19/8,$$

whereas, if we take the height equal to the height at the right-hand corner we get 31/8. The area is really 3, as we find by § 60. Taking still smaller pieces the result is of course better; thus with 30 pieces

each $\frac{1}{10}$ unit wide,* the left-hand heights give 2.855, the right-hand heights 3.155. With still more numerous (smaller) pieces these approximate results approach the true value of the area. (See § 67, p. 116.)

EXERCISES XXIV. STEP-BY-STEP SUMMATION — APPROXIMATE RESULTS

1. Approximate to about 1 % the areas under the curves below, between the limits indicated. Estimate the answers roughly in advance. Use judgment with regard to scales to gain in accuracy by having the figure as large as is convenient. Check results by integration where possible.

(*a*) $y = 1 + x^2$; $x = 0$ to 3.

(*b*) $y = \dfrac{x^3}{100}$; $x = 5$ to 10.

(*c*) $y = x^2 - 2x$; $x = 1$ to 3.

(*d*) $y = 4x^2 - x^4$; $x = 0$ to 2.

(*e*) $y = x^{-2}$; $x = 1$ to 10.

(*f*) $y = x^{-1}$; $x = 10$ to 100.

(*g*) $y = (1 + x)/x$; $x = 2$ to 4.

(*h*) $y = \sqrt{9 + x}$; $x = 0$ to 7.

(*i*) $y = \sqrt{9 + x^2}$; $x = 0$ to 4.

(*j*) $y = \sqrt{9 + x^4}$; $x = 0$ to 2.

2. Approximate to about 1% the distance passed over between the indicated time limits, where the speed is as below; when possible check by integration.

(*a*) $v = 1 + \sqrt{t}$; $t = 0$ to 100.

(*b*) $v = 2t + t^2$; $t = 1$ to 4.

(*c*) $v = \dfrac{1}{1 + \sqrt{t}}$; $t = 0$ to 100.

(*d*) $v = \dfrac{1}{2t + t^2}$; $t = 1$ to 4.

(*e*) $v = \dfrac{1 + t^2}{t^2}$; $t = 1$ to 10.

(*f*) $v = \dfrac{t^2}{1 + t^2}$; $t = 1$ to 10.

(*g*) $v = \dfrac{1 + \sqrt{t}}{\sqrt{t}}$; $t = 4$ to 9.

(*h*) $v = \dfrac{\sqrt{t}}{1 + \sqrt{t}}$; $t = 4$ to 9.

(*i*) $v = \dfrac{1 - t}{1 + t}$; $t = 0$ to 50.

(*j*) $v = \sqrt[3]{1 + t^2}$; $t = 10$ to 20.

3. The volume of a metal casting is often found by dividing the entire pattern into parts, each of which can be computed readily. Show how to find the volume of a flat casting shaped like the letter H, if the thickness and the width of each portion is given.

* In this computation the formula $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(2n + 1)(n + 1)/6$ is convenient. For any reasonable degree of accuracy, this method, in this example, is longer than that of § 60, but for other examples, especially when the curve is drawn and we know no equation for it, this method is often convenient. Notice that the average of the two last results found above is reasonably accurate; it is 3.005 ft. (See § 66, p. 114.)

4. Show how to calculate approximately the volume of a dumbbell whose ends are spheres. Notice that a small volume at the intersection of the spheres with the cross-bar is neglected.

5. Show how to find the volume of a cone approximately, by adding together layers perpendicular to its axis.

6. Find the volume of a sphere by imagining it divided into small pyramids with their vertices at the center and their bases in the surface, as in elementary geometry.

7. Discuss the approximate evaluation of areas in a plane by counting the squares in a figure drawn on cross-section paper. Would still more finely ruled paper be more accurate? Show that the area of any closed figure may be *defined* by extending this process indefinitely.

8. The volume of a ship is computed by means of the areas of cross sections at small distances from each other; show how the result is calculated. Show how to make a more accurate computation by the same method.

9. In shipments of ores or coal, it is usual to sample each car; show how to obtain the total amount of metal in a shipment of several car-loads of ore. Is the result accurate or approximate? Show how a more accurate result can be found.

10. The number of bacteria in a river is computed by sampling at various distances from the shore. Show that the total thus computed is reasonably accurate, on the assumption that the bacteria per cubic foot is approximately constant for short distances.

11. The total sales of a given stock or bond in one year on the New York Stock Exchange can be computed from the record of the number sold each day and the price on that day. Show that the result lies between that found by using the highest and the lowest daily prices. Would the average of the latter two be more accurate?

12. The number in 100,000 persons alive at any given age who die before they are one year older is important in life insurance; show how to compare the actual death rate of a given group of people, — say of the students in a given university, — with the published figures showing the normal expectation of death during each year of age.

13. The amount of cément used in concrete varies in different portions of the same building from one part in two to one part in six. Show how to find the entire amount of cement used in the work from the specifications.

66. Exact Results. Summation Formula. As in the preceding articles, given the rate of increase $R(x)$ of a variable quantity y we can always compute the total difference in the values of y between two values of x, $x=a$ and $x=b$, [*i.e.* the integral $\int_{x=a}^{x=b} R(x)\,dx$]:

Let us break up the interval $x=a$ to $x=b$ into n portions, each of size Δx; the first interval is from a to $a+\Delta x$, the second from $a+\Delta x$ to $a+2\,\Delta x$, and so on. The change in y during each interval can be computed approximately by taking the rate of change as constant and equal to its value at the beginning of the interval; doing so we would obtain, in the first interval a change $\Delta x \cdot R(a)$; in the second $\Delta x \cdot R(a+\Delta x)$; in the third $\Delta x \cdot R(a+2\,\Delta x)$; etc., so that the total change is (approximately) the sum:

$$(1)\quad s = \Delta x \cdot R(a) + \Delta x \cdot R(a+\Delta x) + \Delta x \cdot R(a+2\,\Delta x) + \cdots + \Delta x \cdot R[a+(n-2)\,\Delta x] + \Delta x \cdot R[a+(n-1)\,\Delta x].$$

If we take the constant rate as the rate at the end of the interval, we get the sum

$$(2)\quad S = \Delta x \cdot R(a+\Delta x) + \Delta x \cdot R(a+2\,\Delta x) + \Delta x \cdot R(a+3\,\Delta x) + \cdots + \Delta x \cdot R[a+(n-1)\,\Delta x] + \Delta x \cdot R(a+n\cdot\Delta x).$$

The first of these sums contains the term $\Delta x \cdot R(a)$, the second the term $\Delta x \cdot R(a+n\cdot\Delta x)$; their difference is $D = S - s = \Delta x \cdot [R(a+n\cdot\Delta x) - R(a)] = \Delta x[R(b) - R(a)]$ since $b = a + n\cdot\Delta x$. If, for example, the rate $R(x)$ is increasing, the correct answer evidently lies *between* S and s. S is too high, s is too low. As we make the intervals smaller and more numerous, Δx will approach zero, n will become infinite,* and $D = S - s = \Delta x[R(b) - R(a)]$ will approach zero, since $R(b) - R(a)$ is a constant.

Hence it is evident that *the correct value of the total change in y is the limit of the sum s* (or of the sum S, since the difference between S and s approaches zero); *that is:*

* For the meaning of this phrase, see § 14, p. 19.

$$(3)\quad \int_{x=a}^{x=b} R(x)\,dx = \lim_{\Delta x \doteq 0}\Big\{\Delta x \cdot R(a) + \Delta x \cdot R(a+\Delta x) + \cdots + \Delta x \cdot R[a+(n-1)\Delta x]\Big\}.$$

This formula will be called the **Summation Formula of the Integral Calculus.**

Interpreted as a *motion problem*, $R(x)$ means the *speed*, x denotes time, y distance; the intervals Δx are small intervals of time during which we conceive the speed as sensibly constant; $\Delta x \cdot R(a)$ is the distance (approximately) traversed in the first interval, during which the speed is supposed to remain approximately equal to the speed $R(a)$ at the beginning of the interval; and so on, as in the example of § 65.

Graphically, if x and y denote any concrete quantities one pleases, drawing x horizontally as usual, we may represent the rate $R(x)$ by a

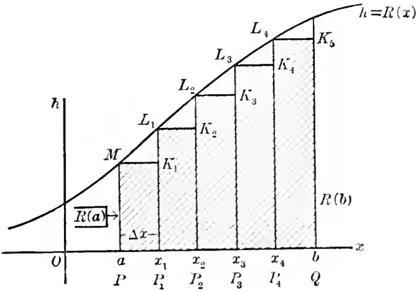


FIG. 28.

curve whose height is h: $h = R(x)$. The intervals Δx are small intervals along the x-axis; PP_1, P_1P_2, P_2P_3, $\cdots$, in each of which we think of the height $h = R(x)$ as sensibly constant. The product $\Delta x \cdot R(a)$ is the area PP_1K_1M of the rectangle whose base is Δx and whose height is $PM = h]_{x=a} = R(a)$. The next term of the sum s is $\Delta x \cdot R(a+\Delta x)$, which is the area of the rectangle $P_1P_2K_2L_1$, and so on; the whole sum s is the area of the polygon $PQK_5L_4K_4L_3K_3L_2K_2L_1K_1M$ in Fig. 28, in which n is taken equal to 5.

Likewise the sum S is the area (Fig. 29) of the polygon $PQNJ_5L_4J_4L_3J_3L_2J_2L_1J_1$, which is exterior to the curve.

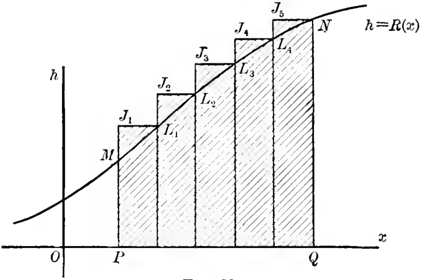


FIG. 29.

The difference $D = S - s = \Delta x\,[R(b) - R(a)]$ is the area of a rectangle whose base is Δx and whose altitude $[R(b) - R(a)]$ is the difference between PM and QN; and it is evident that this area approaches zero with Δx.

The area of either polygon, s or S, evidently approaches the area A under the curve between $x = a$ and $x = b$ as Δx approaches zero:

$$(4)\quad A\Big]_{x=a}^{x=b} = \lim_{\Delta x \doteq 0} S = \lim_{\Delta x \doteq 0} s = \lim_{\Delta x \doteq 0} \{\Delta x\, R(a) + \cdots + \Delta x\, R[a + (n-1)\Delta x]\}$$

which agrees with our previous formulas since

$$(5)\qquad A\Big]_{x=a}^{x=b} = \int_{x=a}^{x=b} h\,dx = \int_{x=a}^{x=b} R(x)\,dx,$$

and the two values of A agree, by (3). This agreement may be regarded, however, as a new proof of (3), since the two formulas for A are obtained independently; but attention is called to the fact that this argument is simply a special case of the general argument used above.

It is evident from the figures that (3) holds also if $R(x)$ is decreasing, or indeed even if $R(x)$ changes from increasing to decreasing, or conversely.

67. Integrals as Limits of Sums. By far the greater number of integrations appear more naturally as *limits of sums* than as *reversed rates.*

Thus, as a matter of fact, even the area A under a curve, treated in § 60 as a reversed rate, probably appears more *naturally* as the limit of a sum, as in (4), § 66. Of course the two are equivalent, since (3), § 66, is true; in any case the results are **calculated** always either **approximately**, as in the exercises under § 65, or else **precisely** by the methods of §§ 58–59. Hence the method of § 59 was given first, because it is used for each calculation even when the problem arises by a summation process.

On account of the frequent occurrence of the summation process, we may say that *an integral really means* a limit of a sum*, but when absolutely precise results are wanted *it is calculated as a reversed differentiation.*† The symbol $\int$ is really a large S somewhat conventionalized, while the dx of the symbol is to remind us of the Δx which occurs in the step-by-step summation.

68. Water Pressure. As another typical instance, consider the water pressure on a dam or on any container.

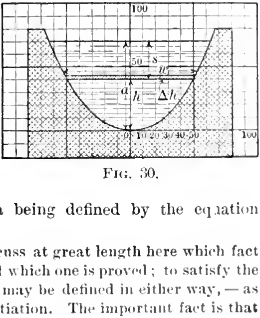


FIG. 30.

The pressure in water increases directly with the depth s, and is equal in all directions at any point. The pressure p on unit area is

(1) $$p = k \cdot s$$

where s is the depth and k is the weight per cubic unit (about 62.4 lb. per cubic foot).

Suppose water flowing in a parabolic channel (Fig. 30), the parabola being defined by the equation

* It is really a waste of time to discuss at great length here which fact about integrals is used as a definition, and which one is proved; to satisfy the demand for formal definition the integral may be defined in either way, — as a limit of a sum, or as a reversed differentiation. The important fact is that the two ideas coincide, which is the fact stated in the Summation Formula.

† Later we return to *approximate* methods of calculation. (See § 125.)

$$(2) \qquad w^2 = 225\,h,$$

where w is the width and h is the height above the bottom. Let a be the total depth of water in the channel. Then the depth s at any point is $s = a - h$, and the pressure is

$$(3) \qquad p = k\,(a - h).$$

If the water is stopped by a cut-off gate, the total pressure on the gate is most easily computed by dividing the gate into horizontal strips of height Δh each; throughout one of the strips the pressure is very nearly constant; the total pressure on a strip is (approximately) the product of its area and the pressure per unit area:

$$(4) \qquad \text{pressure on each strip} = \{w \cdot \Delta h\} \cdot \{p\} = p \cdot w \cdot \Delta h,$$

so that the total pressure P on the gate is (approximately)

$$(5)\ P = \left\{ \Big[p \cdot w \Big]_{h=\Delta h} \cdot \Delta h + \Big[p \cdot w \Big]_{h=2\,\Delta h} \cdot \Delta h + \cdots + \Big[p \cdot w \Big]_{h=n\,.\,\Delta h} \cdot \Delta h \right\}$$

where n is the number of strips. The exact value is therefore

$$(6) \quad P\Big]_{h=0}^{h=a} = \lim_{\Delta h \doteq 0} \left\{ \Big[pw \Big]_{h=\Delta h} \cdot \Delta h + \Big[pw \Big]_{h=2\,\Delta h} \cdot \Delta h + \cdots \right.$$
$$\left. + \Big[pw \Big]_{h=n.\Delta h} \cdot \Delta h \right\} = \int_{h=0}^{h=a} pw\,dh,$$

by (3), p. 115. In the problem before us, $w = 15\,h^{1/2}$ and $p = k(a - h)$; hence

$$(7)\ P\Big]_{h=0}^{h=a} = \int_{h=0}^{h=a} 15\,k(a-h)h^{1/2}\,dh = 15\,k\left[\frac{ah^{3/2}}{3/2} - \frac{h^{5/2}}{5/2}\right]_{h=0}^{h=a} = 4\,ka^{5/2},$$

that is, the total pressure P on the gate increases as the fifth power of the square root of the total depth a of water in the channel; *e.g.* four times the depth of water would mean 32 times the pressure.

Note that the formulas (3), (4), (5), (6) apply in any similar example.

It is important to notice that the total pressure up to any height $h = h$ is a function of h whose *rate of change* is $p \cdot w$. Thus, if the gate be made in two parts, the lower portion, of height h, bears a pressure

$$P_h = P\Big]_{h=0}^{h=h} = 30\,k\left[\frac{ah^{3/2}}{3} - \frac{h^{5/2}}{5}\right]_{h=0}^{h=h} = 30\,k\left(\frac{ah^{3/2}}{3} - \frac{h^{5/2}}{5}\right).$$

The rate of change of P_h as h increases is $dP_h/dh = 15\,k(a-h)h^{1/2} = p \cdot w$.

In general, if the height of the lower portion of the gate be increased by an amount Δh, the pressure P_h on the portion is increased by an amount

$\Delta P_h = p \cdot w\, \Delta h$, approximately, so that $\Delta P_h / \Delta h = p \cdot w$ (nearly) and $dP_h/dh = p \cdot w$ where p is the pressure at the upper edge of the lower portion. The integral in (6) may be thought of as the reversal of this rate, as in §§ 64, 66.

This argument is, however, by no means so natural as the above argument by summation. The important thing to notice is that even in this case the integrated function is really the rate of increase of P as a function of h. But in some problems it is difficult to show directly that the integral is a reversed rate, except by using (3). The great value of the summation formula (3), § 66, is that it makes it unnecessary for us to express each problem as a reversed rate.

EXERCISES XXV. — INTEGRALS AS LIMITS OF SUMS

Determine the following quantities, (a) approximately by step-by-step summation; (b) exactly by integration between limits:

1. The area under the curve $y = x^2$ from $x = 1$ to $x = 3$; from $x = a$ to $x = b$.

2. The area under the curve $y = x^3$ from $x = 0$ to $x = 2$; from $x = -1$ to $x = +1$.

3. The area under the curve $x^2 y = 1$ from $x = 2$ to $x = 5$.

4. The distance passed over by a body whose speed is $v = 2t + 10$ from $t = 0$ to $t = 3$.

5. The distance passed over by a falling body ($v = gt$) from $t = 2$ to $t = 5$.

6. The increase in speed of a falling body from the fact that the acceleration is $g = 32.2$, from $t = 0$ to $t = 3$.

7. The increase in the speed of a train which moves so that its acceleration is $j = t/100$, between the times $t = 0$ and $t = 3$. The distance passed over by the same train, starting from rest, during the same interval of time.

8. The number of revolutions made in 5 min. by a wheel which moves with an angular speed $\omega = t^2/1000$ (radians per second).

9. The time required by the wheel of Ex. 8 to make the first ten revolutions.

10. Repeat Ex. 8 for a wheel for which $\omega = 100 - 10t$ (degrees per second). Find the time required for the first revolution after $t = 0$; note that the speed is decreasing

11. Find the total pressure in tons on one side of the gate of a dry dock, the wet area of the gate being a rectangle 80 ft. long and 30 ft. deep.

12. The pressure in pounds on one side of a board 10 ft. long and 2 ft. wide, which is submerged vertically in water with the upper end 10 ft. below the surface.

13. The pressure on an equilateral triangle 20 ft. on a side, submerged in water with its plane vertical and one side in the surface.

14. The pressure on one side of a square tank 10 ft. high and 5 ft. on a side, the tank being filled with a liquid of specific gravity .8.

15. The pressure on one face of a square 10 ft. on a side, submerged so that one diagonal is vertical and one corner in the surface.

16. The pressure on one end of a parabolic trough filled with water, the depth being 3 ft. and the width across the top 4 ft.

17. The pressure to 1% on a circular disk 10 ft. in diameter, submerged below water with its plane vertical and its center 10 ft. below the surface.

18. The weight of a vertical column of air 1 ft. in cross section and 1 mi. high, given that the weight of air per cubic foot at a height of h feet is $.0805 - .00000268\,h$ pounds.

69. Volumes. — The volumes of many solids may be computed readily by the summation process, either approximately, as in § 65, or exactly by using the Summation Formula, which leads to a reversal of a differentiation. We proceed to illustrate this application by examples.

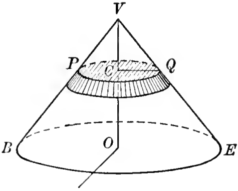


FIG. 31.

Example. To find the volume of a right circular cone whose height h is 10 ft. and the radius of whose base $a = 4$ ft.

Let s be the distance from the vertex V to any plane PQ parallel to the base of the cone. The section of the cone by this plane is a circle whose radius $r = CQ$ is as/h, since the triangles VOE and VCQ are similar; hence the area A_S of this circular section is:

$$(1) \qquad A_S = \pi r^2 = \frac{\pi a^2 \cdot s^2}{h^2}.$$

If we divide up the whole solid into layers of thickness Δs, the volume of each layer is, approximately, the product of its thickness Δs times the area A_S of the bottom of the layer:

(2) $$\textit{Volume of one layer} = V_S = A_S \Delta s;$$

since the value of s at the bottom of the first layer is Δs, the value of s at the bottom of the second layer is $2\Delta s$, etc., the total volume is, approximately,

(3) $$A_S\Big]_{s=\Delta s} \cdot \Delta s + A_S\Big]_{s=2\Delta s} \cdot \Delta s + \cdots + A_S\Big]_{s=n\cdot\Delta s} \cdot \Delta s,$$

where n is the number of layers. Therefore the total volume is

(4) $$V\Big]_{s=0}^{s=h} = \lim_{\Delta s \doteq 0}\left\{A_S\Big]_{s=\Delta s} \cdot \Delta s + A_S\Big]_{s=2\Delta s} \cdot \Delta s + \cdots + A_S\Big]_{s=n\cdot\Delta s} \cdot \Delta s\right\}$$
$$= \int_{s=0}^{s=h} A_S\, ds,$$

by the Summation Formula (3), p. 115. Substituting the value of A_S from (1), we find:

(5) $$V\Big]_{s=0}^{s=h} = \int_{s=0}^{s=h} A_S\, ds = \int_{s=0}^{s=h} \frac{\pi a^2 s^2}{h^2}\, ds = \left[\frac{\pi a^2}{h^2}\frac{s^3}{3}\right]_{s=0}^{s=h} = \frac{\pi a^2 h}{3},$$

which agrees with the formula of ordinary geometry. In this problem, the given values are $h = 10$, $a = 4$; hence, we find $V = 167.5$ cu. ft. (nearly).

Notice that it is quite true that the rate of increase of V as a function of s is $\pi a^2 s^2/h^2$; in general an increase in height Δs causes an increase in volume $A_S \cdot \Delta s$ (nearly) where A_S is the area of the bottom of the layer added; hence $dV/ds = A_S$, which agrees with the integral formula (4), as in §§ 66–68.

70. Volume of Any Frustum. A solid which is bounded at two extremities by a pair of parallel planes is called, in general, a **frustum.***

If such a frustum be divided up into layers of thickness Δs, as in § 69, by planes parallel to the base, and if A_S represents the area of any section at a distance s from the upper bounding

* In special cases, a frustum may touch one or both of the bounding parallel planes in a single point; such special cases include, for example, the sphere; see Ex. 1 below. The two parallel planes which bound the solid are called truncating planes.

plane,* the formulas of § 69 numbered (2), (3), (4) all hold, the arguments being unchanged. If the area A_s is known in terms of s, say $A_s = f(s)$, (4) becomes

$$V\Big]_{s=0}^{s=h} = \int_{s=0}^{s=h} A_s\, ds = \int_{s=0}^{s=h} f(s)\, ds\,;$$

this formula will be called the **Frustum Formula**. It may be used to find the volume of *any* solid, if we know how to find the areas of any complete set of parallel cross sections.

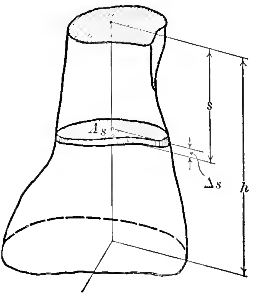


FIG. 32.

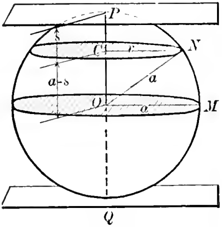


FIG. 33.

Example 1. To find the volume of a sphere of radius a.

The sphere may be thought of as located between two parallel tangent planes at a distance $2a$ from each other. A section parallel to one of the planes is a circle of radius $r = CN$; its area is

$$A_s = \pi r^2\,;$$

but, in the triangle OCN,

$$OC = a - s,\quad ON = a,$$

whence $$r^2 = \overline{CN}^2 = \overline{ON}^2 - \overline{OC}^2 = a^2 - (a-s)^2 = 2as - s^2.$$

It follows that

* In any case, s may be counted from the lower bounding plane, if convenient.

$$V\Big]_{s=0}^{s=2a}=\int_{s=0}^{s=2a}A_S\,ds=\int_{s=0}^{s=2a}\pi(2\,as-s^2)\,ds=\pi\Big[as^2-\frac{s^3}{3}\Big]_{s=0}^{s=2a}=\frac{4\,\pi a^3}{3}.$$

This is, of course, the usual formula; notice that the volume of a hemisphere results by taking the limits $s=0$, and $s=a$, or also by taking $s=a$ and $s=2\,a$. The volume of any frustum of a sphere may be obtained by substituting the correct values of s for the limits of integration; thus the portion of a sphere cut off by a plane at a distance of $a/2$ from the center is

$$V\Big]_{s=0}^{s=a/2}=\int_{s=0}^{s=a/2}\pi(2\,as-s^2)\,ds=\pi\Big[as^2-\frac{s^3}{3}\Big]_{s=0}^{s=a/2}=\frac{5}{24}a^3\pi=\frac{5}{32}\Big(\frac{4}{3}a^3\pi\Big),$$

that is, 5/32 of the volume of the whole sphere.

Example 2. To find the volume of the solid formed by revolving the curve $y=x^2$ about the y-axis; from the vertex to the point where $y=4$.

The solid described is contained between the parallel planes $y=0$ and $y=4$. The section A_S at any height h is a circle; its area is

$$A_S=\pi r^2,$$

where r is the radius of the section. But h and r stand for values of y and x, respectively; hence $h=r^2$ and

$$A_S=\pi x^2=\pi y.$$

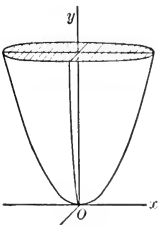


FIG. 34.

Applying the frustum formula, we have

$$V\Big]_{y=0}^{y=4}=\int_{y=0}^{y=4}\pi x^2\,dy=\int_{y=0}^{y=4}\pi y\,dy=\frac{\pi y^2}{2}\Big]_{y=0}^{y=4}=8\,\pi.$$

In general, the **volume formed by revolving any curve** $y=f(x)$ **about the y-axis** between two planes at heights a and b is

$$V\Big]_{y=a}^{y=b}=\int_{y=a}^{y=b}\pi x^2\,dy,$$

where x^2 must be replaced by its value in terms of y from the equation of the curve.

Similarly **the volume of a solid of revolution** formed by revolving the curve $y=f(x)$ **about the x-axis** between the planes $x=a$ and $x=b$ is

$$V\Big]_{x=a}^{x=b}=\int_{x=a}^{x=b}\pi y^2\,dx=\int_{x=a}^{x=b}\pi\,[f(x)]^2\,dx.$$

EXERCISES XXVI. — VOLUMES OF SOLIDS. FRUSTA

1. Find the volume of a frustum of a cone of height h, if the radii of the two bases are, respectively, a and b.

2. Find the volume of the paraboloid of revolution formed by revolving $y^2 = 4x$ about the x-axis, between $x = 0$ and $x = 4$; between $x = 1$ and $x = 5$; between $x = a$ and $x = b$.

3. Find the volume of a hemisphere, using layers parallel to each other.

4. Find the volume of the ellipsoid of revolution formed by revolving an ellipse (1) about its major axis; (2) about its minor axis.

5. Find the volume of the portion of the hyperboloid of revolution formed by revolving about the y-axis the portion of the hyperbola $x^2 - y^2 = 1$ between $y = 0$ and $y = 2$.

6. Find the volume of the portion of the hyperboloid of revolution formed by revolving $x^2 - y^2 = 1$ about the x-axis, between $x = 1$ and $x = 3$.

7. Find the volumes formed by revolving each of the following curves about the x-axis, between $x = 0$ to $x = 2$; between $x = -1$ to $x = +1$:

(*a*) $y = x^3$. (*c*) $y = x^3 - x$. (*e*) $y^2 = x + 2y$.
(*b*) $y = x^2 - 1$. (*d*) $y = (1 + x)^2$. (*f*) $\sqrt{x+1} + \sqrt{y} = 4$.

8. Proceed as in Ex. 7 for each of the following curves, between $x = 1$ and $x = 3$; between $x = a$ and $x = b$:

(*a*) $y = \dfrac{1 + x^2}{x^2}$. (*b*) $xy = 1 + x^2$. (*c*) $x^4 - x^2y^2 = 1$.

9. Proceed as in Ex. 7 for each of the following curves, revolved, however, about the y-axis, between $y = 0$ and $y = 2$:

(*a*) $x = y^3$. (*c*) $x = 4y^2 - y^3$. (*e*) $x = y^2 - y$.
(*b*) $x^2 = y^3$. (*d*) $x^2 + y^4 = 81$. (*f*) $x = y^{1/2} + y^{1/4}$.

10. Find the volume generated when the segment of a parabola from its vertex to its focus revolves (1) about the tangent at the vertex; (2) about the latus rectum.

11. Find the volume generated when the area between the parabola $y = 6x - x^2$ and the x-axis revolves about the x-axis.

12. Find the volume generated when the area bounded by the curves $y = x^2$ and $y^2 = x$ revolves about the x-axis.

13. Calculate the volume of a parabolic trough 10 ft. long, 3 ft. deep, and 4 ft. wide at the top.

14. Find the volume generated by a square of variable size perpendicular to the x-axis, which moves from $x = 0$ to $x = 5$, if the length of the side of the square is (1) proportional to x; (2) equal to x^2.

15. Find the volume generated by a variable equilateral triangle perpendicular to the x-axis, which moves from $x = 0$ to $x = 2$, if a side of the triangle is (1) equal to x^2; (2) proportional to $2 - x$.

16. Find the volume generated by a variable circle which moves in a direction perpendicular to its own plane through a distance 10, if the radius varies as the cube of the distance from the original position.

17. Find the mass of a right circular cylinder of variable density, if the density varies (1) directly as the distance from the base; (2) inversely as the square root of the distance from the base.

71. Cavalieri's Theorem. The Prismoid Formula. If two solids contained between the same two parallel planes have all their corresponding sections parallel to these planes equal, *i.e.* if the area A'_s of such a section for the first solid is the same as the area A''_s of the second, it follows from § 70 that their total volumes are equal, since the two volumes are given by the same integral.

This fact, known as **Cavalieri's Theorem**, is often useful in finding the volumes of solids.

If the area A_s of any section of a frustum is a *quadratic function* of s:*

$$(1) \qquad A_s = as^2 + bs + c$$

where, as in § 70, s represents the distance of the section A_s from one of the two parallel truncating planes, the volume is

$$(2) \qquad V\Big]_{s=0}^{s=h} = \int_{s=0}^{s=h} (as^2 + bs + c)\, ds = \left[a\frac{s^3}{3} + b\frac{s^2}{2} + cs \right]_{s=0}^{s=h}$$

$$= \frac{ah^3}{3} + \frac{bh^2}{2} + ch,$$

where h is the total height of the frustum.

* It is shown in Ex. 3, p. 128, that the results of this section hold also when A_s is any cubic function of s: $A_s = as^3 + bs^2 + cs + d$. Notice also that any linear function $bs + c$ is a special case of (1), for $a = 0$.

The area B of the base of the frustum, the area T of the top, and the area M of a section midway between the top and bottom are

$$B = A_s\Big]_{s=0} = \Big[as^2 + bs + c\Big]_{s=0} = c;$$

$$T = A_s\Big]_{s=h} = \Big[as^2 + bs + c\Big]_{s=h} = ah^2 + bh + c;$$

$$M = A_s\Big]_{s=h/2} = \Big[as^2 + bs + c\Big]_{s=h/2} = a\frac{h^2}{4} + b\frac{h}{2} + c.$$

If we take the *average* of B, T, and 4 times M:

$$\frac{B + T + 4M}{6} = \frac{ah^2}{3} + \frac{bh}{2} + c,$$

this average section multiplied by the total height h turns out to be *exactly* the entire volume:

$$\text{(3)} \qquad \frac{\boldsymbol{B + T + 4M}}{\boldsymbol{6}} \times \boldsymbol{h} = \frac{ah^3}{3} + \frac{bh^2}{2} + ch = \boldsymbol{V}\Big]_{s=0}^{s=h}.$$

This fact is known as the **Prismoid Formula**. It is easy to see by actually checking through the various formulas, *that this formula holds for every solid whose volume is given in elementary geometry;* the same formula *holds for a great variety of other solids.** But the chief use to which the formula is put is for practical approximate computation of volumes of objects in nature: it is reasonably certain that any hill, for example, can be approximated to rather closely either by a frustum of a cone, or of a sphere, or of a cylinder, or of a pyramid, or of a paraboloid; since the prismoid formula holds for all these frusta, it is quite safe to

* The formula holds also, for example, for any *prismoid, i.e.* for a solid with any base and top sections whatever, with sides formed by straight lines joining points of the base to points of the top section. For example, any wedge, even if the base be a polygon or a curve, is a prismoid. The solids defined by (1) include all these and many others; for example, spheres and paraboloids, which are *not* prismoids. The formula holds for all these solids and even (see Ex. 3, p. 128) for all cases where A_s is any cubic function of s. One advantage of the formula is that it is easy to remember: even the formula for the volume of a sphere is most readily remembered by remembering that the prismoid formula holds.

use the formula *without even troubling to see which* of these solids actually approximates to the hill. Similar remarks apply to many other solids, such as metal castings, though it may be necessary to use the formula several times on separate portions of such a complicated object as the pedestal of a statue, or a large bell with attached support and tongue.

Example 1. In the frustum of a paraboloid computed in Ex. 2, p. 123, it is only necessary to notice that the formula for any section

$$A_S = \pi y$$

is a quadratic function of the distance y from one of the two parallel containing planes; indeed, comparing with (1) we see that $a = 0$, $b = \pi$, $c = 0$, so that this case is such an extremely simple "quadratic" that it actually reduces to a linear function, since $a = 0$. Since this results favorably, the prismoid formula applies. It is easy to see that

$$B = 0,\quad T = 4\,\pi,\quad M = 2\,\pi\,;$$

hence $$V = \frac{B + T + 4\,M}{6}\cdot h = \frac{0 + 4\,\pi + 8\,\pi}{6}\cdot 4 = 8\,\pi$$

which agrees with the result of Ex. 2, § 70.

Example 2. The prismoid formula applies to any frustum of an ellipsoid of revolution cut off by planes perpendicular to the axis of revolution.

Let the origin be situated on one of the truncating planes of the frustum, and let the axis of x be the axis of revolution. Then the equation of the generating ellipse is of the form $Ax^2 + By^2 + Dx + F = 0$. The area A_S of a section parallel to the bases is πy^2, since the section is a circle whose radius is y. Hence

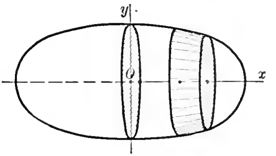


FIG. 35.

$$A_S = \pi y^2 = \pi\left(-\frac{A}{B}x^2 - \frac{D}{B}x - \frac{F}{B}\right),$$

which is a quadratic function of the distance x from one of the truncating planes of the frustum. Therefore the prismoid formula holds.

Beware of applying the prismoid formula, as anything but an approximation formula, without knowing that the area of a section is a quadratic function of s, or (Ex. 3, p. 128) a cubic function of s.

EXERCISES XXVII. — GENERAL EXERCISES

[This list includes a number of exercises which are intended for reviews.]

1. Show that the prismoid formula holds for each of the following elementary solids; hence calculate the volume of each of them by that formula: (*a*) sphere; (*b*) cone; (*c*) cylinder; (*d*) pyramid; (*e*) frustum of a sphere; (*f*) frustum of a cone. See *Tables*, II, F.

2. Calculate the volume of the solid formed by revolving the area between the curve $y = x^2$ and the x-axis about the x-axis, between $x = 0$ and $x = 2$. Find the same volume (approximately) by the prismoid formula, and show that the error is about 4.2 %.

3. Calculate the volume of a frustum of a solid bounded by planes $h = 0$ and $h = H$, if the area A_S of a parallel cross section is a cubic function $ah^3 + bh^2 + ch + d$ of the distance h from one base, first by direct integration, then by the prismoid formula. Hence prove the statement of the footnote, p. 125.

4. In which of the exercises under Exs. 4–9, List XXVI, does the prismoid formula give a precise answer?

5. How much is the percentage error made in computing the volume in Ex. 8 *a*, List XXVI, from $x = 1$ to $x = 3$, by use of the prismoid formula?

6. Show, by analogy to § 71, that the area under any curve whose ordinate y is any quadratic function (or any cubic function) of x, between $x = a$ and $x = b$, is

$$\frac{(b-a)}{6}[y_A + 4\,y_M + y_B],$$

where y_A, y_B, y_M represent the values of y at $x = a$, $x = b$, $x = (a + b)/2$, respectively.

7. Calculate, first by direct integration, and then by the rule of Ex. 6, the areas under each of the following curves:

(*a*) $y = x^2 + 2x + 3$ between $x = 1$ and $x = 5$.
(*b*) $y = x^2 + px + q$ between $x = a$ and $x = b$.
(*c*) $y = x^3 + 5x$ between $x = 2$ and $x = 4$.

8. Calculate approximately the area under the curve $y = x^4$ between $x = 1$ and $x = 3$ by the rule of Ex. 6. Show that the error is about .55 %.

9. Show that the area under the curve $y = 1/x^2$ between $x = 1$ and $x = 5$ can be found more accurately from the rule of Ex. 6 by first dividing the area into two parts with equal bases.

10. Show that any integral whose integrand $f(x)$ is a quadratic (or a cubic) function of x, can be evaluated by a process analogous to the prismoid rule:

$$\int_{x=a}^{x=b} f(x)\,dx = \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\frac{b-a}{6}.$$

11. Evaluate the integral $\int (1/x^2)\,dx$ between $x = 1$ and $x = 5$ approximately, first by the rule of Ex. 10; then by applying the same rule twice in intervals half as wide; then by applying the rule to intervals of unit width.

12. Show that any integral $\int f(x)\,dx$ can be computed approximately by using Ex. 10 with an even number of intervals of small width Δx:

$$\int_{x=a}^{x=b} f(x)\,dx = \Big[f(a) + 4f(a+\Delta x) + 2f(a + 2\,\Delta x) + 4f(a + 3\,\Delta x)$$
$$+ \cdots + f(b)\Big]\frac{\Delta x}{3}.$$

[This rule is called Simpson's Rule; see § 125.]

13. Calculate the following integrals approximately by the process suggested in Exs. 11–12. Notice that some of them cannot be evaluated otherwise at present:

(*a*) $\int_0^3 x^5\,dx.$ (*c*) $\int_0^2 \sqrt{x}\,dx.$ (*e*) $\int_0^1 \sqrt{1+x^2}\,dx.$

(*b*) $\int_1^3 (1/x)\,dx.$ (*d*) $\int_1^5 \sqrt{1+x}\,dx.$ (*f*) $\int_0^{\pi/4} \sin x\,dx.$

14. Show that the **area of any surface of revolution**, formed by revolving a curve $y = f(x)$ about the x-axis, is the limit of a sum of terms of the form $2\pi y \Delta s$, where s denotes the length of arc, as in § 61. Hence show, by § 61, that the area is given by the integral

$$2\pi \int y\,ds = 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

15. In a manner analogous to Ex. 14, show that the area of a surface of revolution formed by revolving a curve about the y-axis is $2\pi \int x\,ds$.

16. Find approximately the length of the arc of the curve $y = x^2$ from $x = 0$ to $x = \frac{1}{2}$; from $x = \frac{1}{2}$ to $x = 1$. (See Ex. 1, p. 108.)

17. Find approximately the area of the convex surface of that portion of the paraboloid formed by revolving the curve $y = \sqrt{x}$ about the x-axis which is cut off by the planes $x = 0$ and $x = \frac{1}{2}$; by $x = \frac{1}{2}$ and $x = 1$.

CHAPTER VI

TRANSCENDENTAL FUNCTIONS

PART I. LOGARITHMS — EXPONENTIAL FUNCTIONS

72. Necessity of Operations on Transcendental Functions. The necessity for the introduction of transcendental functions in the Calculus depends not only on their own general importance, but also upon the fact that *integrals of algebraic functions may be transcendental.*

Thus, in § 57, in the case $n = -1$ the integral $\int x^n\,dx$ could not be found, although the integrand $1/x$ is comparatively simple. We shall see that this integral, $\int x^{-1}\,dx$, results in a *logarithm.* (See § 78, p. 137, Ex. 3.) We shall see also in § 81 that numerous cases arise in science in which the rate of variation of a function $f(x)$ is precisely $1/x$.

In Ex. 1, p. 108, the integral $\int\sqrt{1+4x^2}\,dx$ could not be evaluated; throughout Chapter V, integrals involving radicals were avoided except in special cases, because such integrals usually result in transcendental functions.

73. Properties of Logarithms. The **logarithm** L of a number N to any base B is defined by the fact that the two equations

(1) $$N = B^L, \qquad \log_B N = L$$

are equivalent. Thus if $L = \log_B N$ and $l = \log_B n$, the identity $B^L \cdot B^l = B^{L+l}$ is equivalent to the rule

(2) $$\log_B(N \cdot n) = \log_B N + \log_B n,$$

where n and N are any two numbers. Likewise $B^L \div B^l = B^{L-l}$ gives

(3) $$\log_B(N \div n) = \log_B N - \log_B n;$$

and $(B^L)^n = B^{Ln}$ becomes

(4) $$\log_B N^n = n \log_B N,$$

where n may have any value whatever.

Another fundamental rule results from the application of (4) to the equation

(5) $$x = B^y, \quad i.e. \quad y = \log_B x.$$

For if b is any other base,

(6) $$\log_b x = \log_b (B^y) = y \log_b B; \qquad \text{[by (4)]}$$

but since $y = \log_B x$, we have

(7) $$\log_b x = \log_B x \cdot \log_b B.$$

In particular if $x = b$, since $\log_b b = 1$, we have

(8) $$1 = \log_B b \cdot \log_b B, \text{ or } \log_b B = 1 \div \log_B b.$$

The equations (1), (2), (3), (4), (7), (8) are the fundamental rules for logarithms. (See *Tables*, II, A.)

74. Graphical Representation. A fairly accurate graph of the equation

(1) $$y = \log_B x$$

is obtained by writing the equation in the form

(2) $$x = B^y,$$

and plotting a few points given by taking integral (positive and negative) values of y. Thus $y = 0, 1, 2, \cdots, -1, -2, \cdots$ give $x = 1, B, B^2, \cdots, 1/B, 1/B^2, \cdots$. The student should draw a figure from such values, for several different values of B, taking $B = 2$, then $B = 3, 5, 10$, etc. When $B = 1$, the equation (2) degenerates into the horizontal straight line $x = 1$, while (1) degenerates completely and becomes meaningless; for this reason, *the number 1 is never used as a base of logarithms.*

To make these graphs accurately, more points are necessary. The easiest method is to calculate the desired values by common logarithms, *i.e.* logarithms to the base 10. Taking the common logarithms of both sides of (2), we find

$$\log_{10} x = \log_{10} B^y = y \cdot \log_{10} B,$$

(3) or $$y = \log_{10} x \div \log_{10} B.$$

It should be noticed that (3) is equivalent to (1) and therefore to (2); the curves for $B = 1.5$, $B = 2$, $B = 3$, $B = 4.5$, $B = 9$ are shown in the figure. They should be carefully drawn on a much larger scale by the student, by use of (3). See *Tables.*

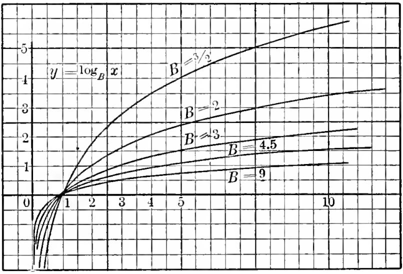


FIG. 36.

EXERCISES XXVIII. — LOGARITHMS AND EXPONENTIALS

1. Find the value of 10^x when $x = 2$; 0; 1.5; 2.3; -1; -1.7; 0.43.

2. Plot the curve $y = 10^x$ carefully, using several fractional values of x.

3. Plot the curve $y = \log_{10} x$ by direct comparison with the figure of Ex. 2. Plot it again by use of a table of logarithms.

4. Plot the graph of each of the following functions:

(*a*) $\log_{10} x^2$. (*b*) $\log_{10} (1/x)$. (*c*) $\log_{10} \sqrt{x}$. (*d*) $\log_{10} x^{2/3}$.

Do any relations exist between these graphs?

5. Plot the graph of each of the following functions and explain its relation to graphs already drawn above:

(*a*) $\log_{10} (1 + x)$. (*b*) $\log_{10} \sqrt{1 + x}$. (*c*) $\log_{10} (x\sqrt{1 + x})$.

6. Plot the graphs of each of the following functions and show the relations between them.

(*a*) $\log_2 x$. (*b*) $\log_4 x$. (*c*) $\log_2 x^2$. (*d*) 2^x.

7. Show how to calculate most readily the values of the following expressions, and find the numerical value of each one:

(*a*) $\log_{11} 7$. (*c*) $(5.4)^{6.2}$. (*e*) $10^{1.5} + 10^{-1.5}$. (*g*) $\log_5 10$.

(*b*) $2^{4.53}$. (*d*) $\log_5 8$. (*f*) $5 \log_4 5$. (*h*) $10^{\log_{10} 7}$.

8. Draw each of the following curves:

(*a*) $y = 10^x + 10^{-x}$. (*c*) $y = x \log_{10} x$. (*e*) $y = \log_{10} \cos x$.

(*b*) $pv^{1.41} = \text{const.}$ (*d*) $y = 2^x \sin x$. (*f*) $y = 10^{\sin x}$.

75. Slope of $y = \log_{10} x$ at $x = 1$. The slope M of the curve

(1) $$y = \log_{10} x$$

at the point (1, 0) can be approximated very closely. Let (1, 0) be called P, and let $(1 + \Delta x, 0 + \Delta y)$ be called Q; then

$$0 + \Delta y = \log_{10} (1 + \Delta x),$$

and the slope m_{PQ} of PQ is

(2) $$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{\log_{10}(1 + \Delta x)}{\Delta x}.$$

If Δx is given in succession the values .1, .01, .001, we find

$$m_{PQ}\Big]_{\Delta x = .1} = 10 \log_{10} (1.1) = 0.4139;$$

$$m_{PQ}\Big]_{\Delta x = .01} = 100 \log_{10} (1.01) = 0.432;$$

$$m_{PQ}\Big]_{\Delta x = .001} = 1000 \log_{10} (1.001)$$

$$= 0.43 \text{ [using five-place tables]}$$

$$= 0.434 \text{ [using six- or seven-place tables]}.$$

Still smaller values of Δx would give the same result by the usual interpolation rules, so that for values of Δx less than .001 a table of more than seven places would be needed; and even then the result would be changed at most in the fourth place of decimals.

The slope M of the curve (1) at (1, 0) is the limit of these slopes as Δx approaches zero; hence

(3) $$M = \frac{dy}{dx}\Big]_{x=0} = \lim_{\Delta x \doteq 0} m_{PQ} = 0.434 \cdots \text{ (approximately).*}$$

76. Differentiation of $\log_{10} x$. It is now easy to find the derivative of $\log_{10} x$. Let P, (x, y), be any point for which

(1) $$y = \log_{10} x, \quad \text{or} \quad x = 10^y;$$

and let Q, $(x + \Delta x,\ y + \Delta y)$, be any other point on the curve; then

(2) $$y + \Delta y = \log_{10}(x + \Delta x), \quad \text{or} \quad x + \Delta x = 10^{y+\Delta y}.$$

Subtracting the second form of (1) from the second form of (2),

$$\Delta x = 10^{y+\Delta y} - 10^y = 10^y(10^{\Delta y} - 1)$$

and

(3) $$\frac{\Delta x}{\Delta y} = 10^y \cdot \frac{(10^{\Delta y} - 1)}{\Delta y}, \quad \text{or} \quad \frac{\Delta y}{\Delta x} = \frac{1}{x} \cdot \frac{\Delta y}{10^{\Delta y} - 1},$$

since $x = 10^y$.

In particular at $x = 1$,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{10^{\Delta y} - 1},$$

which, by § 75, approaches the limit $M = 0.434 \cdots$.

In general,† therefore,

(4) $$\frac{dy}{dx} = \frac{d \log_{10} x}{dx} = \lim_{\Delta x \doteq 0}\left[\frac{1}{x} \cdot \frac{\Delta y}{10^{\Delta y} - 1}\right] = \frac{M}{x} = \frac{0.434 \cdots}{x}.$$

* This assumes only that the ordinary interpolation scheme for common logarithms is approximately correct. The number M is so important that its value has been calculated to a large number of decimal places; to ten places it is 0.4342944819. *An independent method of calculating it is* given in § 134. Logically, the present approximate determination of M could be omitted entirely until that time, and M could be carried through all the work as an unknown constant. Practically, it is very desirable to have an approximate value of M at once.

† The difficulties ordinarily met in proving this formula are here avoided by placing the burden of any difficulty where it should be, — upon the read-

77. Differentiation of $\log_B x$. Since by (3), § 74, the equation

(1) $$y = \log_B x$$

can also be written in the form

(2) $$y = \log_{10} x \div \log_{10} B,$$

it follows that

(3) $$\frac{dy}{dx} = \frac{d \log_B x}{dx} = \frac{d \log_{10} x}{dx} \div \log_{10} B = \frac{M}{x} \div \log_{10} B.$$

Since the number M which occurs in all these formulas is an inconvenient decimal, it is useful to find a value of B, for which

(4) $$\log_{10} B = M = 0.434 \cdots;$$

this value is readily found from a logarithm table, and is denoted by the letter e:

(5) $$\boldsymbol{e = 10^M = 2.72 \cdots} \textbf{ (approximately)}.$$

If $B = e$, the formula (3) becomes

[VIII$_a$] $$\frac{d \log_e x}{dx} = \frac{1}{x}.$$

*On account of the simplicity of this formula the base e will be used henceforth in this book for all logarithms and exponentials unless the contrary is explicitly stated;** it is called the **natural base**, or the **Napierian base.**

If B has any value whatever, (3) becomes

[VIII] $$\frac{d \log_B x}{dx} = \frac{1}{x} \cdot \frac{M}{\log_{10} B} = \frac{1}{x} \frac{\log_{10} e}{\log_{10} B} = \frac{1}{x} \cdot \log_B e;$$

ing of an ordinary table of logarithms; for the essence of the difficulty lies in the lack of accuracy of the usual elementary definition of logarithms. No pretense of rigorous logic in the proof of (4) is justified unless a proof that the common logarithm of any number exists is given.

* The value of e to ten places is 2.7182818285. Another method of computing its value is given in § 134; see also § 142.

for theoretical purposes, the last form is used; for practical computations, the next to the last.

If $B = 10$, we find

$$[\text{VIII}_b] \qquad \frac{d \log_{10} x}{dx} = \frac{M}{x} = \frac{\log_{10} e}{x} = \frac{0.434 \cdots}{x}.$$

These three Rules, of which [VIII] is the general form, are added to the list of seven Rules in Chapter III. While the common base 10 is exceedingly convenient for computations, the new base e is simpler in all theoretical discussions, chiefly because [VIII_a] is simpler than [VIII_b].

Logarithms to the base e are called *natural*, or *Napierian*, or *hyperbolic* logarithms. See *Tables*, V, C.

78. Illustrative Examples. We may now combine Rule [VIII] with [I]–[VII], and with the reverse differentiation (integration) formulas of Chapter V.

Example 1. Given $y = \log_{10}(2x^2 + 3)$, to find dy/dx.

Method 1. Derivative notation. Set $u = 2x^2 + 3$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d \log_{10} u}{du} \cdot \frac{d(2x^2+3)}{dx} = \frac{M}{u} \cdot 4x = \frac{4Mx}{2x^2+3}.$$

Method 2. Differential notation.

$$dy = d \log_{10}(2x^2+3) = \frac{M}{2x^2+3} d(2x^2+3) = \frac{4Mx}{2x^2+3} dx.$$

Example 2. Find the area under the curve $y = 1/x$ from $x = 1$ to $x = 10$:

$$A\Big]_{x=1}^{x=10} = \int_{x=1}^{x=10} \frac{1}{x} dx = \log_e x \Big]_{x=1}^{x=10} = \log_e 10 = \frac{1}{\log_{10} e} = \frac{1}{M} = 2.3026.*$$

* The number $\log_e 10 = 1 \div M = 2.302585$ is important because common logarithms (base 10) are reduced to natural logarithms (base e) by multiplying by this number, since $\log_e N = \log_{10} N \times \log_e 10$. Similarly, natural logarithms are reduced to common logarithms by multiplying by $M = \log_{10} e$; since $\log_{10} N = M \div \log_e N$. It is easy to remember which of these two multipliers should be used in transferring from one of these bases to the other by remembering that logarithms of numbers above 1 are surely greater when e is used as base than when 10 is used.

Example 3. If the rate of increase dy/dx of a quantity y with respect to x is $1/x$, find y in terms of x.

Since $dy/dx = 1/x$,

$$y = \int \frac{1}{x}\,dx = \log_e x + c,$$

where c is a constant, — the value of y when $x = 1$. It should be noted that logarithms to the base e occur here in a perfectly natural manner; the same remark applies in Example 2. Note that $\log_e x = \log_{10} x \div M$.

This case arises constantly in science. Thus, if a volume v of gas expands by an amount Δv, and if the work done in the expansion is ΔW, the ratio $\Delta W/\Delta v$ is approximately the pressure of the gas; and $dW/dv = p$ exactly. If the temperature remains constant $pv =$ a constant; hence $dW/dv = k/v$. The general expression for W is therefore

$$W = \int \frac{k}{v}\,dv = k \log_e v + c,$$

and the work done in expanding from one volume v_1 to another volume v_2 is

$$W\Big]_{v=v_1}^{v=v_2} = \int_{v=v_1}^{v=v_2} \frac{k}{v}\,dv = k \log_e v\Big]_{v_1}^{v_2} = k \log_e \frac{v_2}{v_1} = \frac{k}{M} \log_{10} \frac{v_2}{v_1}.$$

EXERCISES XXIX. — LOGARITHMS

1. Calculate the derivative of each of the following functions; when possible, simplify the given expression first:

(*a*) $\log_{10} x^2$. (*b*) $\log_{10} \sqrt{x}$. (*c*) $\log_{10} (1 + 3x)$.

(*d*) $\log_{10} (1 + x^2)$. (*e*) $\log_e (1 + x)^2$. (*f*) $\log_e \sqrt{1 + 2x}$.

(*g*) $\log_e (1/x)$. (*h*) $\log_{10} (x^{-2})$. (*i*) $x \log_e x$.

(*j*) $\log_e \left(\dfrac{1+x}{1-x}\right)$. (*k*) $\log_{10} \left(2 + \dfrac{t}{1+t}\right)$. (*l*) $\log_e \sqrt{\dfrac{t}{1-t}}$.

(*m*) $\dfrac{\log_e t}{t}$. (*n*) $\log_e \{\log_e x\}$. (*o*) $(\log_e t)^3$.

2. Evaluate each of the following integrals:

(*a*) $\displaystyle\int_1^2 \frac{3}{x}\,dx$. (*b*) $\displaystyle\int_3^4 \frac{1+x}{x}\,dx$. (*c*) $\displaystyle\int_5^6 \frac{2 - 3x^2 + x^4}{x^3}\,dx$.

(*d*) $\displaystyle\int_{10}^{20} \frac{1+x^2}{x}\,dx$. (*e*) $\displaystyle\int_{10}^{100} \frac{(2-t)^3}{3t^4}\,dt$. (*f*) $\displaystyle\int_1^e \frac{6t^5 - 2t^2 - 1}{3t^3}\,dt$.

(*g*) $\displaystyle\int_1^2 \frac{x^{1/2} - 1}{x^{3/2}}\,dx$. (*h*) $\displaystyle\int_1^4 \frac{s^{3/2} - 2}{s^{5/2}}\,ds$. (*i*) $\displaystyle\int_5^{10} (1 - u^{-1})(1 + u^{-2})\,du$

3. Calculate the area between the hyperbola $xy = 1$ and the x-axis, from $x = 1$ to 10, 10 to 100, 100 to 1000; from $x = 1$ to $x = k$.

4. Show that the slope of the curve $y = \log_{10} x$ is a constant times the slope of the curve $y = \log_e x$. Determine this constant factor.

5. Find the flexion of the curve $y = \log_e x$, and show that there are no points of inflexion on the curve.

6. Find the maxima and minima of the curve $y = \log_e (x^2 - 2x + 3)$.

7. Find the maxima and minima and the points of inflexion (if any exist), on each of the following curves:

(*a*) $y = 2x^2 - \log_e x$. (*b*) $y = x + \log_e (1 + x^2)$.
(*c*) $y = x^3 - \log_e x^3$. (*d*) $y = (2x + \log x)^2$.

8. Find the areas under each of the following curves between $x = 2$ and $x = 5$:

(*a*) $y = x + 1/x$. (*b*) $y = (x^2 + 1)/x^3$. (*c*) $y = (x^{1/2} - x)/x^2$.

9. Find the volume of the solid of revolution formed by revolving that portion of the curve $xy^2 = 1$ between $x = 1$ and $x = 3$ about the x-axis. How much error would be made in calculating this volume by the prismoid formula?

10. If a body moves so that its speed $v = t + 1/t$, calculate the distance passed over between the times $t = 2$ and $t = 4$.

11. Find the work done in compressing 10 cu. ft. of a gas to 5 cu. ft., if $pv = .004$.

12. Find the areas under the hyperbola $xy = k^2$ between $x = 1$ and $x = c$, c and c^2, c^2 and c^3, c^3 and c^4.

79. Differentiation of Exponentials. Since the equations

$$y = \log_B x \text{ and } x = B^y$$

are equivalent, Rule [VIII] gives

$$\frac{dx}{dy} = \frac{dB^y}{dy} = 1 \div \frac{dy}{dx} = \frac{x}{\log_B e} = B^y \cdot \log_e B.$$

If we interchange the letters x and y, for convenience of memory, we obtain the standard forms:

$$\boldsymbol{y = B^x} \qquad (\text{or } x = \log_B y)$$

$$\textbf{[IX]} \qquad \frac{dy}{dx} = \frac{d\,\boldsymbol{B^x}}{dx} = \frac{y}{\log_B e} = \boldsymbol{B^x \cdot \log_e B};$$

of which the two special cases $B = e$ and $B = 10$ are:

[IX$_a$] $$\frac{dy}{dx} = \frac{de^x}{dx} = y = e^x.$$

[IX$_b$] $$\frac{dy}{dx} = \frac{d\,10^x}{dx} = \frac{y}{M} = \frac{10^x}{M} = 10^x(2.3026).$$

This formula [IX] can be combined with all the preceding rules, as in § 78.

80. Illustrative Examples.

Example 1. Given $y = e^{x^2}$, to find dy/dx.

Method 1. Set $x^2 = u$; then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{de^{x^2}}{du} \cdot \frac{d(x^2)}{dx} = 2\,x \cdot e^u = 2\,x\,e^{x^2}.$$

Method 2. $$dy = de^{x^2} = e^{x^2}\,d(x^2) = 2\,x\,e^{x^2}\,dx.$$

Example 2. Find the length l of the arc of the *catenary* $y = (e^x + e^{-x})/2$, between the points where $x = 0$ and where $x = 1$.

By § 61, p. 107, we have

$$l\Big]_{x=0}^{x=1} = \int_{x=0}^{x=1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx, \quad \left[\frac{dy}{dx} = \frac{e^x - e^{-x}}{2}\right]$$

$$= \int_{x=0}^{x=1} \sqrt{1 + \frac{(e^x - e^{-x})^2}{4}}\,dx = \frac{1}{2}\int_{x=0}^{x=1} (e^x + e^{-x})\,dx$$

$$= \frac{1}{2}\Big[e^x - e^{-x}\Big]_{x=0}^{x=1} = \frac{1}{2}(e^1 - e^{-1}) - \frac{1}{2}(1 - 1)$$

$$= \frac{1}{2}\left(e - \frac{1}{e}\right) = (2.718 - 0.368)/2 = 1.175 \text{ (nearly)}.$$

This curve is very important because it is the form taken by a perfect inelastic cord hung between two points. The given function is often called the **hyperbolic cosine** of x, and is denoted by **cosh** x, so that $\cosh x = (e^x + e^{-x})/2$.

Example 3. If a quantity y has a rate of change dy/dx with respect to x proportional to y itself, to find y in terms of x. Given

$$\frac{dy}{dx} = ky,$$

we may write

$$k\frac{dx}{dy}=\frac{1}{y},$$

hence

$$kx=\int\frac{1}{y}\,dy=\log_e y+c,$$

by § 78, Ex. 3. Transposing c, we have

$$\log_e y = kx - c, \quad \text{or} \quad y = e^{kx-c} = e^{-c}e^{kx} = Ce^{kx},$$

where $C(=e^{-c})$ is again an arbitrary constant.

The only quantity y whose rate of change is proportional to itself is Ce^{kx} where C and k are arbitrary, and k is the factor of proportionality. This principle is of the greatest importance in science; a detailed discussion of concrete cases is taken up in § 81.

EXERCISES XXX. — EXPONENTIALS

1. Show that the slope of the curve $y=e^x$ is equal to its ordinate.

2. Show that the area under the curve $y=e^x$ between the y-axis and any value of x is $y-1$.

3. Find the derivative of each of the following functions:

(*a*) e^{2x}. (*b*) e^{x^2}. (*c*) xe^x.

(*d*) $(e^x+1)^2$. (*e*) $\dfrac{e^x+e^{-x}}{2}$. (*f*) $\dfrac{e^x-e^{-x}}{2}$.

(*g*) $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$. (*h*) $\dfrac{e^{-x^2}}{1+x^2}$. (*i*) x^2e^{3x}.

(*j*) e^{3x^2+4}. (*k*) 10^{x^2}. (*l*) $x\cdot 10^{2x+3}$.

4. The expression $(e^x-e^{-x})/2$, used in Ex. 3 (*f*) is called the **hyperbolic sine** of x; and $(e^x+e^{-x})/2$ is called the **hyperbolic cosine** of x; they are represented by the symbols **sinh** x and **cosh** x respectively. See *Tables*, II, H. Show that

$$d\sinh x=\cosh x\,dx, \quad d\cosh x=\sinh x\,dx.$$

5. Show that $1+\sinh^2 x=\cosh^2 x$; hence find the length of the arc of the curve $y=\cosh x$ from $x=0$ to $x=2$.

[The curve $y=\cosh x$, or $y=(e^x+e^{-x})/2$, is called a **catenary** (§ 80).]

6. Find the area under the catenary from $x=0$ to $x=3$; from $x=-1$ to $x=+1$; from $x=0$ to $x=a$. [See *Tables*, V, C.]

7. Find the area under the curve $y=\sinh x$ from $x=0$ to $x=3$; from $x=0$ to $x=a$.

8. Find the maxima and minima and the points of inflexion (if any exist) on each of the following curves:

(*a*) $y = \sinh x$. (*b*) $y = \cosh x$. (*c*) $y = \tanh x = \sinh x/\cosh x$.

(*d*) $y = e^{-x^2}$. (*e*) $y = e^{-x^4}$. (*f*) $y = \operatorname{sech} x = 1 \div \cosh x$.

9. Show that the pair of parameter equations $x = \cosh t$, $y = \sinh t$ represent the rectangular hyperbola $x^2 - y^2 = 1$. Hence show that the differential of arc for this hyperbola is $ds = (\cosh 2t)^{1/2}\,dt$, and find the speed at the point where $t = 0$, if t denotes the time.

10. Show that the area under the hyperbola $x^2 - y^2 = 1$ from $x = 1$ to $x = a$ is represented by the integral

$$\int_{t=0}^{t=c} \sinh^2 t\,dt = \int_{t=0}^{t=c} [(\cosh 2t - 1)/2]\,dt$$

where $\cosh c = a$. Hence show that this area is $(\sinh 2c)/4 - c/2$.

11. Show that the area of a triangle whose vertices are the origin, the point $(x, 0)$, the point (x, y) on the hyperbola $x^2 - y^2 = 1$, is $xy/2 = (\sinh 2t)/4$. [Ex. 9.] Hence show by Ex. 10 that the portion of this triangle outside of the hyperbola is $t/2$.

[Note. The parameter equations are often written in the form $x = \cosh 2A$, $y = \sinh 2A$, where A is the last area mentioned.]

12. Calculate the following integrals:

(*a*) $\int_0^2 e^x\,dx$. (*d*) $\int_0^1 \sinh 2x\,dx$. (*g*) $\int_1^{10} (e^x + 1)^2\,dx$.

(*b*) $\int_0^1 e^{-x}\,dx$. (*e*) $\int_1^2 \cosh 3x\,dx$. (*h*) $\int (e^x + 3)\,e^{-x}\,dx$.

(*c*) $\int_1^4 e^{2x}\,dx$. (*f*) $\int_0^5 \sinh^2 x\,dx$. (*i*) $\int (e^{2x+3} + 1)\,dx$.

81. Compound Interest Law. The fact proved in the Ex. 3 of § 80 is of great importance in science:

If a variable quantity y has a rate of increase

$$\frac{dy}{dx} = ky \tag{1}$$

with respect to an independent variable x proportional to y itself, then

$$y = Ce^{kx}, \tag{2}$$

where C is an arbitrary constant.

For this reason the equation (2) between two variables x and y was called by Lord Kelvin the "**Compound Interest Law,**" on account of its crude analogy to compound interest on money. For the larger the amount y (of principal and interest) grows the faster the interest accumulates.

"Compound interest" is, however, only a convenient name, since interest is really compounded at stated intervals (*e.g.* each year) and *not* continuously. A more suggestive name might be **the snowball law,** since a snowball grows more rapidly the larger it becomes, and its rate of growth is roughly proportional to its size.

In science instances of a rate of growth which grows as the total grows are frequent.*

Example 1. *Work in Expanding Gas.* The example used to illustrate Ex. 3, § 78, can be put in this form. Since, in the work W done in the expansion at constant temperature of a gas of volume v, we found $dW/dv = k/v$, it follows that $dv/dW = v/k$; hence $v = Ae^{W/k}$, which agrees with the result of § 78.

Example 2. *Cooling in a Moving Fluid.* If a heated object is cooled in running water or moving air, and if θ is the varying difference in temperature between the heated object and the fluid, the rate of change of θ (per second) is assumed to be proportional to θ; $d\theta/dt = -k\theta$, where t is the time and where the negative sign indicates that θ is decreasing. It follows that $\theta = C \cdot e^{-kt}$. [Newton's Law of Cooling.]

Such an equation may also be thrown in the form of § 78; in this example, $dt/d\theta = -1/(k\theta)$, whence $t = -(1/k) \cdot \log_e \theta + c$, and the time taken to cool from one temperature θ_1 to another temperature θ_2 is

$$t\Big]_{\theta=\theta_1}^{\theta=\theta_2} = \int_{\theta_1}^{\theta_2} -\frac{d\theta}{k\theta} = -\frac{1}{k}\log_e \theta\Big]_{\theta_1}^{\theta_2} = -\frac{1}{k}\log_e \frac{\theta_2}{\theta_1},$$

where θ is the temperature of the body above the temperature of the surrounding fluid.

* The common expressions "grows like a snowball," "gathers momentum as it goes," "wealth breeds wealth," "it grows by its very growth," "the rich grow richer, the poor poorer" illustrate the frequent occurrence of such cases.

The law for the dying out of an electric current in a conductor when the power is cut off is very similar to the law for cooling in this example. See Ex. 17, p. 146.

Example 3. *Bacterial Growth.* If bacteria grow freely in the presence of unlimited food, the increase per second in the number in a cubic inch of culture is proportional to the number present. Hence

$$\frac{dN}{dt} = kN, \quad N = Ce^{kt}, \quad t = \frac{1}{k}\log_e N + c,$$

where N is the number of thousand per cubic inch, t is the time, and k is the rate of increase shown by a colony of one thousand per cubic inch. The time consumed in increase from one number N_1 to another number N_2 is

$$t\Big]_{N_1}^{N_2} = \int_{N_1}^{N_2} \frac{1}{k}\frac{dN}{N} = \frac{1}{k}\log_e N\Big]_{N_1}^{N_2} = \frac{1}{k}\log_e \frac{N_2}{N_1}.$$

If $N_2 = 10\,N_1$, the time consumed is $(1/k)\log_e 10 = 1/(kM)$. This fact is used to determine k, since the time consumed in increasing N tenfold can be measured (approximately). If this time is T, then $T = 1/(kM)$, whence $k = 1/(TM)$, where T is known and $M = 0.43$ (nearly).

Numerous instances similar to this occur in vegetable growth and in organic chemistry. For this reason the equation (2) on p. 141 is often called the "**law of organic growth.**" (See Exs. 18, 19, p. 146.)

Example 4. *Atmospheric Pressure.* The air pressure near the surface of the earth is due to the weight of the air above. The pressure at the bottom of 1 cu. ft. of air exceeds that at the top by the weight of that cubic foot of air. If we assume the temperature constant, the volume of a given amount is inversely proportional to the pressure, hence the amount of air in 1 cu. ft. is directly proportional to the pressure, and therefore the weight of 1 cu. ft. is proportional to the pressure. It follows that the rate of decrease of the pressure as we leave the earth's surface is proportional to the pressure itself:

$$\frac{dp}{dh} = -\,kp, \quad p = Ce^{-kh}, \quad h = -\frac{1}{k}\log_e p + c,$$

where h is the height above the earth; and, as in Exs. 2 and 3, the difference in the height which would change the pressure from p_1 to p_2 is

$$h\Big]_{p_1}^{p_2} = -1/k \cdot \log_e\left(\frac{p_2}{p_1}\right).$$

Since $h\big]_{p_1}^{p_2}$, and p_2 and p_1 can be found by experiment, k is determined by the last equation.

82. Percentage Rate of Increase. The principle stated in § 81 may be restated as follows: In the case of bacterial growth, for example, while the total rate of increase is clearly proportional to the total number in thousands to the cubic inch of bacteria, *the percentage rate of increase* is clearly constant.

In any case the **percentage rate of increase**, r_p, is obtained by dividing 100 times the *total rate of increase by the total amount of the quantity*, $100 \cdot (dy/dx) \div y$; and since the equation $dy/dx = ky$ gives $(dy/dx) \div y = k$, it is clear that the *percentage rate of increase in any of these problems is a constant.* The quotient $(dy/dx) \div y$, that is, 1/100 of the percentage rate of increase, will be called the **relative rate of increase**, and will be denoted by r_r.

In some of the exercises which follow, the statements are phrased in terms of *percentage* rate of increase, r_p, or the *relative* rate of increase, $r_r = r_p \div 100$.

EXERCISES XXXI. — COMPOUND INTEREST LAW

1. If $y = 5\,e^{2x}$, find dy/dx, and show that $(dy/dx) \div y = 2$.

2. Find dy/dx and $(dy/dx) \div y$ for each of the following functions:

(*a*) $7\,e^{3x}$. (*d*) e^{x^2}. (*g*) $(ax + b)e^{kx}$.

(*b*) $4\,e^{-2.5x}$. (*e*) e^{4x+5}. (*h*) $(x^2 + px + q)e^x$.

(*c*) xe^x. (*f*) $(x^2 + 2)e^x$. (*i*) $(3\,x + 2)e^{-x^2}$.

3. If a body cools in moving air, according to Newton's law, $d\theta/dt = -k\theta$, where t is the time (in seconds) and θ is the difference in temperature between the body and the air, find k if θ falls from 40° C. to 30° C. in 200 seconds.

4. How soon will the difference in temperature θ in Ex. 3, fall to 10° C.?

5. If a body is cooled in air, according to Newton's law, find k if the θ changes from 20° C. to 10° C. in five minutes. How soon will θ reach 5° C.?

6. If a body cools so that the percentage rate of cooling is 2 % (in degrees C. and minutes), how long will it take to cool from a difference 20° to a difference 10° (with respect to the surrounding air)?

7. In measuring atmospheric pressure, it is usual to express the pressure in millimeters (or in inches) of mercury in a barometer. Find C in the formula of Ex. 4, § 81, if $p = 762$ mm. when $h = 0$ (sea level). Find C if $p = 30$ in. when $h = 0$.

8. Using the value of C found in Ex. 7, find k in the formula for atmospheric pressure if $p = 24$ in. when $h = 5830$ ft.; if $p = 600$ mm. when $h = 1909$ m. Hence find the barometric reading at a height of 3000 ft.; 1000 m. Find the height if the barometer reads 28 in.; 650 mm.

[NOTE. Pressure in pounds per square inch $= 0.4908 \times$ barometer reading in inches.]

9. If a rotating wheel is stopped by water friction, the rate of decrease of angular speed, $d\omega/dt$, is proportional to the speed. Find ω in terms of the time, and find the factor of proportionality if the speed of the wheel diminishes 50 % in one minute.

10. If a wheel stopped by water friction has its speed reduced at a constant rate of 2 % (in revolutions per second and seconds), how long will it take to lose 50 % of the speed?

11. The length l of a rod when heated expands at a constant rate per cent ($= 100\,k$). Show that $dl/d\theta = kl$, where θ is the temperature; if the percentage rate of increase is .001 % (in feet and degrees C.), how much longer will it be when heated 200° C.? At what temperature will the rod be 1 % longer than it was originally?

[NOTE. This value of k is about correct for cast iron.]

12. The **coefficient of expansion** of a metal rod is the increase in length per degree rise in temperature of a rod of unit length. Show that the coefficient of expansion of any rod is the relative rate of increase in length with respect to the temperature. (See Ex. 12, p. 27.)

13. A chimney is designed so that the pressure per square inch on each horizontal cross section is a constant k. If the outer surface of a section at a height h is a circle of radius R, and if all the cross sections are similar, including the flue holes, show that the total pressure on a cross section is proportional to kR^2, and that $k(R + \Delta R)^2 = kR^2 - \rho R^2 \Delta h$, where ρ is the weight per cubic inch of the material. Hence show that $dR/dh = -\rho R/(2\,k)$ and that $R = R_0 e^{-\rho h/(2k)}$, where R_0 is the radius of the bottom section ($h = 0$).

14. Assuming that the form of a chimney is given by the equation $R = R_0 e^{-\rho h/(2k)}$ [Ex. 13], show that the total weight (neglecting the flue holes) is $k\pi R_0^2(1 - e^{-\rho H/k})$, where H is the total height. Hence show that

the pressure per square inch on the bottom section is $k(1-e^{-\rho H/k})$, and that it approaches the theoretical limit k as H increases.

15. Show that the *results* of Ex. 14 are the same when the flue holes are taken into account, with the assumptions made in Ex. 13.

[NOTE. The pressure per square inch depends solely on the height, for the same material. The height is limited by the crushing strength of the material.]

16. When a belt passes around a pulley, if T is the tension (in pounds) at a distance s (in feet) from the point where the belt leaves the pulley, r the radius of the pulley, and μ the coefficient of friction, then $dT/ds = \mu T/r$. Express T in terms of s. If $T = 30$ lb. when $s = 0$, what is T when $s = 5$ ft., if $r = 7$ ft., and $\mu = 0.3$?

17. When an electric circuit is cut off, the rate of decrease of the current is proportional to the current C. Show that $C = C_0e^{-kt}$, where C_0 is the value of C when $t = 0$.

[NOTE. The assumption made is that the electric pressure, or electromotive force, suddenly becomes zero, the circuit remaining unbroken. This is approximately realized in one portion of a circuit which is short-circuited. The effect is due to **self-induction**: $k = R/L$, where R is the resistance and L the self-induction of the circuit.]

18. Radium automatically decomposes at a constant (relative) rate. Show that the quantity remaining after a time t is $q = q_0e^{-kt}$, where q_0 is the original quantity. Find k from the fact that half the original quantity disappears in 1800 yrs. How much disappears in 100 yrs.? in one year?

19. Many other chemical reactions — for example, the formation of invert sugar from sugar — proceed approximately in a manner similar to that described in Ex. 18. Show that the quantity which remains is $q = q_0e^{-kt}$ and that the amount transformed is $A = q_0 - q = q_0(1-e^{-kt})$. Show that the quantities which remain after a series of equal intervals of time are in geometric proportion.

20. The amount of light which passes through a given thickness of glass, or other absorbing material, is found from the fact that a fixed per cent of the total is absorbed by any absorbing material. Express the amount which will pass through a given thickness of glass.

83. Logarithmic Differentiation. Relative Increase. In § 82 we defined the *relative rate of increase* r_r of a quantity y with respect to x as the total rate of increase (dy/dx) divided by y.

If y is given as a function of x,

(1) $$y = f(x),$$

the relative rate of increase $r_r = (dy/dx) \div y$ can be obtained by taking the logarithms of both sides of (1),*

(2) $$\log_e y = \log_e f(x),$$

and then differentiating both sides with respect to x:

(3) $$r_r = \frac{1}{y} \cdot \frac{dy}{dx} = \frac{d \log_e y}{dx} = \frac{d \log_e f(x)}{dx}.$$

This process is often called **logarithmic differentiation**: *the logarithmic derivative of a function is its relative rate of increase,* r_r, or $1/100$ of its percentage rate of increase.

Example 1. Given $y = Ce^{kx}$, to find $r_r = (dy/dx) \div y$. Taking logarithms on both sides:

$$\log_e y = \log_e C + kx;$$

differentiating both sides with respect to x;

$$r_r = \frac{dy}{dx} \div y = \frac{d \log_e y}{dx} = k.$$

The result of Ex. 3, p. 139, may be restated as follows: the only function of x whose relative rate of change (logarithmic derivative) is constant is Ce^{kx}.

Example 2. Given $y = x^2 + 3x + 2$, to find r_r.

Method 1. $\dfrac{dy}{dx} = 2x + 3$, hence $r_r = \dfrac{dy}{dx} \div y = \dfrac{2x+3}{x^2+3x+2}$.

Method 2. $r_r = \dfrac{dy}{dx} \div y = \dfrac{d \log y}{dx} = \dfrac{d \log (x^2+3x+2)}{dx} = \dfrac{2x+3}{x^2+3x+2}$.

84. Logarithmic Methods. The process of logarithmic differentiation is often used apart from its meaning as a relative rate, simply as a device for obtaining the usual derivative. This is particularly useful in the case of variables raised to variable powers, and it is at least convenient in such other examples as those which follow.

* Since log N is defined only for positive values of N, *all that follows holds only for positive values of the quantities whose logarithms are used.*

Example 1. Given $y = \sqrt{x}$, to find dy/dx.

Method 1. Ordinary Differentiation.

$$\frac{dy}{dx} = \frac{d\sqrt{x}}{dx} = \frac{dx^{1/2}}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\,x^{1/2}}.$$

Method 2. Logarithmic Method.

Since for positive values of $\sqrt{x}$,

$$\log y = \log x^{1/2} = \tfrac{1}{2}\log x,$$

we have $\quad r_r = dy/dx \div y = 1/(2\,x);\ dy/dx = y/(2\,x) = 1/(2\,x^{1/2}).$

Example 2. Given $y = (2\,x^2 + 3)10^{4x-1}$.

Method 1. Ordinary Differentiation.

$$\begin{aligned}\frac{dy}{dx} &= (2\,x^2+3)\frac{d}{dx}(10^{4x-1}) + 10^{4x-1}\frac{d}{dx}(2\,x^2+3)\\ &= (2\,x^2+3)\cdot 4\cdot\frac{1}{M}10^{4x-1} + 10^{4x-1}\cdot 4\,x\\ &= 4\cdot 10^{4x-1}\Big[(2\,x^2+3)/M + x\Big], \text{ where } M = \log_{10} e = 0.434.\end{aligned}$$

Method 2. Logarithmic Method.

Since $\quad \log y = \log(2\,x^2+3) + (4\,x - 1)\log 10,$

we have $\quad r_r = \dfrac{dy}{dx} \div y = \dfrac{4\,x}{2\,x^2+3} + 4\cdot\log 10,$

or $\quad \dfrac{dy}{dx} = y\left[\dfrac{4\,x}{2\,x^2+3} + 4\log 10\right] = 4\cdot 10^{4x-1}\,[x + (2\,x^2+3)\log 10],$

which agrees with the preceding result, since $\log_e 10 = 1/\log_{10} e = 1/M$.

Example 3. Given $y = (3\,x^2 + 1)^{2x+4}$, to find dy/dx. Since no rule has been given for a variable to a variable power, ordinary differentiation cannot be used advantageously. Taking logarithms, however, we find

$$\log y = (2\,x+4)\log(3\,x^2+1),$$

whence $\quad r_r = \dfrac{dy}{dx} \div y = 2\log(3\,x^2+1) + \dfrac{6\,x}{3\,x^2+1}(2\,x+4),$

or $\quad \dfrac{dy}{dx} = (3\,x^2+1)^{2x+4}\left\{2\log(3\,x^2+1) + \dfrac{6\,x}{3\,x^2+1}(2\,x+4)\right\}.$

The use of the logarithmic method is the only expeditious way to find the derivative in this example.

EXERCISES XXXII. — LOGARITHMIC DIFFERENTIATION

1. Find the logarithmic derivatives (relative rates of increase) of each of the following functions, by each of the two methods of §§ 82–83:

(*a*) e^{-2x}. (*e*) $0.1\, e^{10t-5}$. (*i*) $(r^2+1)\, e^{-r^2}$.

(*b*) $4\, e^{4t}$. (*f*) 10^{2x+3}. (*j*) $(2-3\,t^2)\, e^{2t^2-1}$.

(*c*) e^{3t+2}. (*g*) $e^{-x^2+kx^3}$. (*k*) $(1-t^2+t^4)\, 10^{t^3+3t}$.

(*d*) e^{-x^2}. (*h*) $2\,t^2e^{-5t}$. (*l*) e^{e^r}.

2. Find the derivative of each of the following functions by the logarithmic method:

(*a*) $(1+x)^{1+x}$. (*c*) $x^{\sqrt{x}}$. (*e*) $(1+x)(1+2\,x)(1+3\,x)$.

(*b*) $(s^2+1)^{2s+3}$. (*d*) t^{e^t}. (*f*) $\sqrt[3]{1+s^2} \div \sqrt{1-s^2}$.

3. If $y = uv$, show that $dy \div y = du \div u + dv \div v$. In general show that the relative rate of increase of a *product* is the *sum* of the relative rates of increase of the factors.

4. If a rectangular sheet of metal is heated, show that the relative rate of increase in its area is twice the coefficient of expansion of the material [see Ex. 12, List XXXI].

5. Extend the rule of Ex. 3 to the case of any number of factors. Apply this to the expansion of a heated block of metal.

6. Show directly, and also by use of Ex. 5, that the relative rate of increase of x^n with respect to x, where n is an integer, is n/x.

7. Compare the functions e^{2x} and e^{2x+3}; compare their relative rates of increase; compare their derivatives; compare their second derivatives.

8. Compare the following pairs of functions, their logarithmic derivatives, their ordinary derivatives, and their second derivatives:

(*a*) e^x and 10^x. (*d*) e^{-ax} and e^{+ax}.

(*b*) e^{ax} and e^{ax+b}. (*e*) e^{-x^2} and sech x.

(*c*) e^{ax} and 10^{bx}. (*f*) e^{-x^2} and $1 \div (a+bx^2)$.

9. Can k be found so that ke^{ax} and 10^{bx} coincide? Prove this by comparing their logarithmic derivatives, and find b in terms of a.

10. If the logarithmic derivative $(dy/dx) \div y$ is equal to $3+4\,x$, show that $\log y = 3\,x + 2\,x^2 + \text{const.}$, or $y = ke^{3x+2x^2}$.

11. If $(dy/dx) \div y = f(x)$ show that $y = ke^{\int f(x)dx}$.

12. Find y if the logarithmic derivative has any one of the following values:

(*a*) $1-x$. (*c*) n/x. (*e*) e^x.

(*b*) $ax+bx^2$. (*d*) $a+n/x$. (*f*) e^x+n/x.

PART II. TRIGONOMETRIC FUNCTIONS

85. Introduction of Trigonometric Functions. The way in which trigonometric functions enter in the Calculus is illustrated by the following simple case of uniform rotation:

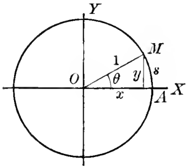


FIG. 37.

A point M moves with a constant speed of 1 ft. per second on a unit circle. Let O be the center of the circle, and let x and y be the horizontal and vertical distances, respectively, of the moving point M from O. The equation of the circle

$$(1) \qquad x^2 + y^2 = 1$$

may also be written in parameter form

$$(2) \qquad x = \cos\theta, \quad y = \sin\theta,$$

where $\theta = \angle XOM$, as is evident from the figure.

If θ is measured in *circular measure*, $\theta = s$, where s is the arc AM, since the radius is 1. Moreover, since M is moving with a constant speed of 1 ft. per second, $s = t$, where t is the time measured in seconds since M was at A, and s is measured in feet. The equation (2) may be written in the form:

$$(3) \qquad x = \cos t, \quad y = \sin t, \ (\text{where } \theta = s = t).$$

The horizontal speed of M, v_x, and its vertical speed, v_y, are, respectively:

$$(4) \qquad v_x = \frac{dx}{dt} = \frac{d\cos t}{dt}, \quad v_y = \frac{dy}{dt} = \frac{d\sin t}{dt};$$

to find these we need precisely to know *the derivatives of* $\cos t$ *and of* $\sin t$ *with respect to* t.

86. Differentiation of Sines and Cosines. These derivatives may be found directly from the example of § 85.

To do so, we need to find *two equations for the two unknown quantities* dx/dt *and* dy/dt; one of these is given by differentiating (1), § 85, with respect to t:

$$x\frac{dx}{dt}+y\frac{dy}{dt}=0; \tag{1}$$

the other is found from the fact that the sum of the squares of v_x and v_y is equal to the square of the total speed (§ 62):

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=1, \tag{2}$$

since the speed $ds/dt=1$ in § 85. Either unknown can now be found by solving (1) and (2) simultaneously:

$$\frac{dx}{dt}=-y=-\sin t,\quad \frac{dy}{dt}=+x=+\cos t, \tag{3}$$

since $x^2+y^2=1$. In extracting square roots in this solution the negative sign is attached to the value of dx/dt because x is decreasing when y is positive. The signs in (3) are easily seen to be correct for both positive and negative values of x and y. Comparing (3) with equation (4) of § 85, we find:

[XI] $\quad \dfrac{d\cos t}{dt}=-\sin t,\qquad$ **[X]** $\quad \dfrac{d\sin t}{dt}=+\cos t,$

or, in differential notation:

[XI]′ $\quad d\cos t=-\sin t\,dt,\qquad$ **[X]′** $\quad d\sin t=+\cos t\,dt.$

*These two formulas are the basis of all work on trigonometric functions. Circular measure of angles was used in obtaining them, and this system of measurement will be used in all that follows.**

A direct proof of these two important formulas is easily made. For, let $y=\sin x$; then $y+\Delta y=\sin(x+\Delta x)$,

and $\qquad \Delta y=\sin(x+\Delta x)-\sin x=2\cos\left(x+\dfrac{\Delta x}{2}\right)\sin\dfrac{\Delta x}{2}.$

Hence $\quad \dfrac{\Delta y}{\Delta x}=\cos\left(x+\dfrac{\Delta x}{2}\right)\cdot\dfrac{\sin(\Delta x/2)}{\Delta x/2},$

* Circular measure of angles is used in the Calculus for the same reason that Napierian Logarithms are used for logarithmic and exponential functions: in each case the standard formulas for differentiation are simplest in the system adopted.

whence $\frac{dy}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \cos x$, or $\frac{d \sin x}{dx} = \cos x$,

since $\lim_{\alpha \doteq 0} (\sin \alpha)/\alpha = 1.$

The proof of [XI] is exactly analogous. See also Ex. 7, p. 154.

87. Illustrative Examples. The formulas [X] and [XI] may be combined with other standard formulas. Some of the results are themselves worthy of mention as new standard formulas; these are numbered below in Roman numerals and printed in black-faced type.

Example 1. Given $y = \sin 2\theta$, find dy.

$$dy = d(\sin 2\theta) = \cos 2\theta d(2\theta) = 2 \cos 2\theta \, d\theta.$$

Example 2. Given $y = \tan \theta$, find $dy/d\theta$.

$$\frac{dy}{d\theta} = \frac{d \tan \theta}{d\theta} = \frac{d \frac{\sin \theta}{\cos \theta}}{d\theta} = \frac{\cos \theta \frac{d \sin \theta}{d\theta} - \sin \theta \frac{d \cos \theta}{d\theta}}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}.$$

[XII] $$\boldsymbol{\frac{d \tan \theta}{d\theta} = \frac{1}{\cos^2 \theta} = \sec^2 \theta.}$$

Example 3. Given $y = \operatorname{ctn} \theta$, to find $dy/d\theta$.

[XIII] $$\boldsymbol{\frac{d \operatorname{ctn} \theta}{d\theta}} = \frac{d \frac{\cos \theta}{\sin \theta}}{d\theta} = -\frac{1}{\sin^2 \theta} = \boldsymbol{-\csc^2 \theta.}$$

Similarly,

[XIV] $$\boldsymbol{\frac{d \sec \theta}{d\theta}} = \frac{d \frac{1}{\cos \theta}}{d\theta} = \frac{\sin \theta}{\cos^2 \theta} = \boldsymbol{\sec \theta \tan \theta.}$$

[XV] $$\boldsymbol{\frac{d \csc \theta}{d\theta}} = \frac{d \frac{1}{\sin \theta}}{d\theta} = \frac{-\cos \theta}{\sin^2 \theta} = \boldsymbol{-\csc \theta \operatorname{ctn} \theta.}$$

Example 4. Given $y = e^x \sin x$, to find dy/dx.

$$\frac{dy}{dx} = \frac{de^x}{dx} \cdot \sin x + \frac{d \sin x}{dx} \cdot e^x = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x).$$

Example 5. Given $y = \cos^3(2t^2 + 1)$, to find dy/dt.

Let $u = 2t^2 + 1$, and $v = \cos u$, then

$$\frac{dy}{dt} = \frac{d(v^3)}{dt} = 3v^2\frac{dv}{dt} = 3v^2\frac{d\cos u}{dt} = -3v^2 \sin u\frac{du}{dt} = -3v^2\sin u\frac{d(2t^2+1)}{dt}$$
$$= -3v^2 \cdot \sin u \cdot 4t = -12t \cdot \sin(2t^2+1)\cdot\cos^2(2t^2+1).$$

Example 6. To find the area under the curve $y = \sin x$ from the point where $x = 0$ to the point where $x = \pi/2$.

$$A\Big]_{x=0}^{x=\pi/2} = \int_{x=0}^{x=\pi/2}\sin x\,dx = -\cos x\Big]_{x=0}^{x=\pi/2} = -\cos\pi/2 + \cos 0 = 1,$$

since $d(-\cos x) = \sin x\,dx$. Comparatively few of the trigonometric integrals can be found by simple inspection; a detailed treatment of them is given in Chapter VII.

EXERCISES XXXIII. — TRIGONOMETRIC FUNCTIONS

1. Find the derivative of each of the following functions:

(*a*) $\sin 3x$. (*e*) $\sin x^2$. (*i*) $x \sin x$.
(*b*) $\cos(\theta/2)$. (*f*) $\tan(2t+3)$. (*j*) $e^t \tan t$.
(*c*) $\tan(-\theta)$. (*g*) $\cos(-4t)$. (*k*) $\log\cos x$.
(*d*) $\cos^2 x$. (*h*) $\sec(x/3)$. (*l*) $\sin e^x$.
(*m*) $\sin x + 3\cos 2x$. (*p*) $e^{-t}\cos^2(1+3t)$.
(*n*) $e^{-t}\sin(2t+\pi/10)$. (*q*) $e^{-2t+1}\sin(3t-\pi/4)$.
(*o*) $(1+x^2)\sin(2x+3)$. (*r*) $e^{-t/10}[\cos t + 4\sin 3t]$.

2. Find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi$; test the correctness of your result by rough comparison with the circumscribed rectangle.

3. Find the area bounded by the two axes and the curve $y = \cos x$, in the first quadrant.

4. Find the maxima and minima, and the points of inflexion (if any exist) on each of the following curves:

(*a*) $y = \sin x$. (*d*) $y = x\sin x$. (*g*) $y = e^{-x}\sin x$.
(*b*) $y = \cos x$. (*e*) $y = 1 + \sin 2x$. (*h*) $y = e^{-2x}\sin x$.
(*c*) $y = \tan x$. (*f*) $y = \sin x + \cos x$. (*i*) $y = \cos(2x + \pi/6)$.

5. Find the derivative of each of the following pairs of functions, and draw conclusions concerning the functions:

(*a*) $\sin x$ and $\cos(\pi/2 - x)$. (*d*) $\sin 2x$ and $2\sin x\cos x$.
(*b*) $\sin^2 x$ and $1 - \cos^2 x$. (*e*) $\cos 2x$ and $2\cos^2 x$.
(*c*) $\cos x$ and $\cos(-x)$. (*f*) $\tan^2 x$ and $\sec^2 x$.

6. Integrate the following expressions; in case the limits are stated, evaluate the integrals, and represent them graphically as areas:

(*a*) $\int_0^{\pi/3} \sin x\,dx.$ (*c*) $\int_0^{\pi/4} \sec^2 x\,dx.$ (*e*) $\int \cos(3t+\pi/6)dt.$

(*b*) $\int_{-\pi/2}^{+\pi/2} \cos x\,dx.$ (*d*) $\int \sin 2x\,dx.$ (*f*) $\int \tan t \sec t\,dt.$

(*g*) $\int_0^{\pi} (1+\sin x)\,dx.$

(*h*) $\int (\cos x + 3\sin 2x)\,dx.$

(*i*) $\int (\cos 2x - 1)\,dx.$

(*j*) $\int \cos^2 x\,dx.$
Hint. $2\cos^2 x = 1 + \cos 2x.$

(*k*) $\int_0^{\pi/2} \sin^2 x\,dx.$

7. Find the derivative of $\sin x$ directly by showing that

$$\sin(x+\Delta x) - \sin x = \sin x(\cos\Delta x - 1) + \cos x \cdot \sin\Delta x$$

and remarking that

$$\lim\,[(\cos\Delta x - 1) \div \Delta x] = 0 \text{ and } \lim\,[(\sin\Delta x) \div \Delta x] = 1.$$

[See § 13, p. 19; Ex. 8, List V; and § 96.]

8. Find the derivative of $\cos x$ directly as in Ex. 7.

9. Find the derivatives of the two functions

(*a*) $\text{vers}\, x = 1 - \cos x.$ (*b*) $\text{exsec}\, x = \sec x - 1.$

10. Differentiate each of the *answers* in the list of formulas, *Tables*, IV, E_a, E_b. What should the result of your differentiation be?

[The teacher will indicate which formulas should be thus tested.]

11. Find the speed of a moving particle whose motion is given in terms of the time t by one of the pairs of parameter equations which follow; and find the path in each case:

(*a*) $\begin{cases} x = 2\cos 3t. \\ y = 2\sin 3t. \end{cases}$ (*c*) $\begin{cases} x = \sin t + \cos t. \\ y = \sin t. \end{cases}$

(*b*) $\begin{cases} x = 2\cos 4t. \\ y = 3\sin 4t. \end{cases}$ (*d*) $\begin{cases} x = \sec t. \\ y = \tan t. \end{cases}$

12. A flywheel 5 ft. in diameter makes 1 revolution per second. Find the horizontal and the vertical speed of a point on its rim 1 ft. above the center.

13. A point on the rim of a flywheel of radius 10 ft. which is 6 ft. above the center has a horizontal speed 20 ft. per second. Find the angular speed, and the total linear speed of a point on the rim.

14. The *cycloid* (Tables, III, G_1) is defined by the equations

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

Find the horizontal and the vertical speeds if t represents the time in the motion of a particle for which these equations hold. Find the total speed; the tangential acceleration. Find the values of each of these quantities when $t = \pi/4$.

15. Find the area of one arch of the cycloid. [See Ex. 6, (*j*)]

16. Show that the differential of the arc, ds, of the cycloid is

$$ds = a\sqrt{2 - 2\cos t}\,dt = 2a\sin(t/2).$$

Hence find the length of one arch of the cycloid.

88. Simple Harmonic Motion. If, as in § 85, a point M moves with constant speed in a circular path, the projection P of that point on any straight line is said to be in **simple harmonic motion.**

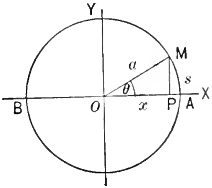


FIG. 38.

Let the circle have a radius a; let the constant speed be v; and let the straight line be taken as the x-axis. We may suppose the center of the circle lies on the straight line, since the projection of the moving point on either of two parallel straight lines has the same motion. Let the center O of the circle be the origin. Then we have

$$(1) \qquad x = OP = a\cos\theta, \quad \text{or} \quad x = a\cos(s/a),$$

where $s = \text{arc } AM$, since $\theta = s/a$. Moreover, since the speed v is constant, $v = s/T$, if T is the time since M was at A; or $v = s/(t - t_0)$ if t is measured from any instant whatever, and t_0 is the value of t when M is at A. We have therefore

$$(2) \qquad x = a\cos\frac{s}{a} = a\cos\left[\frac{v}{a}(t - t_0)\right] = a\cos[kt + \epsilon];$$

where $k = v/a$, and $\epsilon = -kt_0 = -vt_0/a$.

From (2), the speed dx/dt of P along BA is

$$(3) \qquad v_x = \frac{dx}{dt} = \frac{d[a\cos(kt + \epsilon)]}{dt} = -ak\sin(kt + \epsilon),$$

and the acceleration of P is

$$(4)\qquad j_T = \frac{d^2x}{dt^2} = -ak^2\cos(kt+\epsilon) = -k^2\cdot x,$$

or,

$$(5)\qquad j_T \div x = \frac{d^2x}{dt^2} \div x = -k^2;$$

that is, the acceleration of x divided by x, is a negative constant, $-k^2$. We shall see that much of the importance of simple harmonic motion arises from this fact.

It is important to notice that (2) may be written in the form

$$x = a\cos(kt+\epsilon) = a[\cos\epsilon\cos kt - \sin\epsilon\sin kt],$$

or

$$(6)\qquad x = A\sin kt + B\cos kt,$$

where $A = -a\sin\epsilon$ and $B = +a\cos\epsilon$ are both constants. The form (6) may be used to derive (5) directly.

The simplest forms of the equation (6) result when $k=1$ and either $A=0$ and $B=1$, or $A=1$ and $B=0$:

$$(7)\qquad \begin{cases} x = \sin t; \text{ if } k=1,\ A=1,\ B=0,\ \textit{i.e.}\ a=1,\ \epsilon=3\pi/2. \\ x = \cos t; \text{ if } k=1,\ A=0,\ B=1,\ \textit{i.e.}\ a=1,\ \epsilon=0. \end{cases}$$

The formulas (2) and (6) are general formulas for simple harmonic motion; (7) represents two especially simple cases.

89. Relative Acceleration. The ratio of d^2x/dt^2 to x found above is the **relative acceleration** of x.

In Ex. 8, p. 149, we saw that the function $x = ae^{kt+c}$ gave $d^2x/dt^2 = k^2x$, or $(d^2x/dt^2)\div x = k^2$, that is, the relative acceleration of x is a positive constant, k^2.

In § 88, we saw that *a simple harmonic motion*, represented by (2) or (6) of § 88, gives $d^2x/dt^2 \div x = -k^2$; that is, the relative acceleration is a negative constant, $-k^2$. It will appear later (see § 187) that *these are the only functions for which the relative acceleration is a constant different from zero.*

90. Vibration. The importance of simple harmonic motion, based on its property (5) of § 88, is evident in *vibrating bodies*, such as vibrating cords or wires, the prongs of a tuning fork, the atoms of water in a wave, a weight suspended by a spring.

In all such cases, it is natural to suppose that the force which tends to restore the vibrating particle to its central position increases with the distance from that central position, and is proportional to that distance. (Compare Hooke's law in Physics.)

It is a standard law of physics, equivalent to Newton's second law of motion, that the acceleration of any particle is proportional to the force acting upon it. (See § 52, p. 82.)

In the case of vibration, therefore, the acceleration, being proportional to the force, is proportional to the distance, x, from the central position; it follows that, in ordinary vibrations, the relative acceleration is a negative constant, — negative, because the acceleration is opposite to the positive direction of motion. For this reason, each particle of a vibrating body is supposed to have a simple harmonic motion, unless disturbing causes, such as air friction, enter to change the result. Neglecting such frictional effects temporarily, the distance x from the central position is, as in § 88,

$$x = a \cos (kt + \epsilon) = A \sin kt + B \cos kt,$$

where t denotes the time measured from a starting time t_0 seconds before the particle is at $x = a$, and where $\epsilon = - t_0 k$. Moreover, from § 88 and also from what precedes,*

$$\frac{d^2x}{dt^2} = - k^2 x.$$

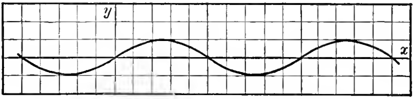
The quantity a is called the *amplitude*, $2\pi/k$ is called the *period*, and $t_0 = - \epsilon/k$ is called the *phase*, of the vibration.

91. Waves. Another important application of S. H. M. is in the treatment of wave motions. Thus the form of a simple vibration of a stretched cord or wire is assumed to be

$$y = a \sin \frac{nx}{l} \pi,$$

* **Electric vibrations** follow this same law if the resistance is negligible. If v represents the electromotive force in volts, $d^2v/dt^2 = -k^2 v$, where k is a constant. The sudden discharge of an electric condenser by a good conductor would give such an electric vibration. But the effect of the electric resistance (which corresponds to the friction in mechanical vibrations) is very marked, and the vibrations die out with extreme rapidity.

where l is the total length of the cord between the fixed ends and n is the number of arches in the wave.



$$y = \sin \frac{nx}{l} \pi \text{ for } l = 5n; \text{ i.e. } y = \sin \frac{\pi x}{5}.$$

FIG. 39.

A compound vibration of such a stretched cord is thought of as made up by combining several such simple vibrations simultaneously:

$$y = a_1 \sin \frac{n_1 x}{l} \pi + a_2 \sin \frac{n_2 x}{l} \pi + \cdots + a_p \sin \frac{n_p x}{l} \pi.$$

An alternating electric current varies with the time in a similar manner; for a simple alternating current,

$$C = a \sin kt,$$

where C is the current in amperes and t is the time measured in seconds from a time when $C = 0$; or the sum of several such terms for a compound current.

In general, a sum of several simple harmonic terms:

$$a_1 \sin (k_1 t + \epsilon_1) + a_2 \sin (k_2 t + \epsilon_2) + \cdots + a_p \sin (k_p t + \epsilon_p)$$

is called a **compound harmonic function.** See *Tables*, III, F.

EXERCISES XXXIV. — SIMPLE HARMONIC MOTION — VIBRATIONS

1. Find the speed and the acceleration of a particle whose displacement x has one of the following values; compare the acceleration with the original expression for the displacement:

(*a*) $x = \sin 2t$.

(*b*) $x = \sin (t/2 - \pi/4)$.

(*c*) $x = \sin t - \frac{1}{2} \sin 2t$.

(*d*) $x = \cos t + \frac{1}{3} \cos 3t$.

(*e*) $x = \sin 2t + 0.15 \sin 6t$.

(*f*) $x = \sin t - \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t$.

(*g*) $x = a \sin (kt + e)$.

(*h*) $x = A \cos kt + B \sin kt$.

2. Determine the angular acceleration of a hair spring if it vibrates according to the law $\theta = .2 \sin 10\pi t$; what is the amplitude of one vibration, the period and the extreme value of the acceleration?

3. Show that each of the following functions satisfies an equation of the form $d^2u/dt^2 + k^2u = 0$ or $d^2u/dt^2 - k^2u = 0$; in each case determine the value of k:

(*a*) $u = 10 \sin 3t$.
(*b*) $u = 0.7 \cos 13t$.
(*c*) $u = 5e^{3t}$.
(*d*) $u = 20e^{-2t}$.
(*e*) $u = \sin(5t + \pi/3)$.
(*f*) $u = 5\cos(t/2 - \pi/12)$.
(*g*) $u = 12\cos 4t - 5\sin 4t$.
(*h*) $u = 3\sin 5t + 4\cos 5t$.
(*i*) $u = C_1 \sin 3t + C_2 \cos 3t$.
(*j*) $u = C_1e^{4t} + C_2e^{-4t}$.

4. Show that the function $u = A\sin kt + B\cos kt$ always satisfies the equation $d^2u/dt^2 + k^2u = 0$ for any values of A and B. Check by substituting various positive and negative values for k, A, B.

5. Show that $u = Ae^{kt} + Be^{-kt}$ always satisfies the equation

$$d^2u/dt^2 - k^2u = 0.$$

6. Substitute $u = e^{mx}$ in the equation $d^2u/dx^2 = 4u$, and show that e^{mx} is a solution if, and only if, $m^2 = 4$. Show that $u = Ae^{2x} + Be^{-2x}$ is a solution for all values of A and B.

7. By the methods of Exs. 4–6, write down by inspection as general a solution as possible for each of the following equations:

(*a*) $\dfrac{d^2x}{dt^2} = -x$.
(*b*) $\dfrac{d^2x}{dt^2} = -4x$.
(*c*) $\dfrac{d^2x}{dt^2} = -\dfrac{1}{4}x$.
(*d*) $\dfrac{d^2x}{dt^2} = 9x$.
(*e*) $\dfrac{d^2x}{dt^2} = 16x$.
(*f*) $\dfrac{d^2x}{dt^2} = 12x$.

[NOTE. Any equation which contains derivatives is called a **differential equation.** Many simple ones have been used. It is shown in Chapter X that the solutions found in Exs. 4–6 are the most general solutions.]

8. The differential equation of falling bodies is $d^2s/dt^2 = -g$; show that $s = -gt^2/2 + C_1t + C_2$. Find C_1 and C_2 if $s = 0$ and the speed $v = 0$ when $t = 0$; if $s = 0$ and $v = 100$, when $t = 0$.

9. The differential equation of a certain vibrating body is $d^2s/dt^2 = -s$; show that $s = A\sin t + B\cos t$; find A and B if $s = 0$ and the speed $v = 10$ when $t = 0$; if $s = 2$ and $v = 0$ when $t = 0$.

10. A flywheel 6 ft. in diameter revolves with a uniform speed of 30 R. P. M. Write the differential equation of the projection on the floor of a point on the rim.

11. A horizontal slider S attached by an exceedingly long connecting rod SP to a pivot P on a wheel whose center is O, is forced to move

approximately as the projection of P on the floor. If the wheel rotates uniformly, show that the distance s of the slider from its central position is approximately $s = a \sin kt$, where $a = OP$ (the "*crank*"), and $k = 2\pi n$, where the wheel revolves n times per second, and OP is vertical when $t = 0$.

12. If OP in Ex. 11 makes an angle ϵ with the vertical when $t = 0$, show that $s = a \sin (kt + \epsilon)$.

[NOTE. a is called the *amplitude*, $n = k/2\pi$ the *frequency*, and ϵ the *phase* (or phase-angle) of the S. H. M. of the slider.]

13. When an electrical condenser discharges through a negligible resistance the current C follows the law $\dfrac{d^2C}{dt^2} = -a^2C$, where a is a constant. Express the current in terms of the time. When $a = 1000$, what is the frequency (number of alternations) per second?

14. Any ordinary alternating electric current varies in intensity according to the law $C = a \sin kt$; find the maximum current and the time-rate of change of the current.

15. Show that two terms of the form $(a \cos kt + b \sin kt)$ and $(A \cos kt + B \sin kt)$ combine into one term of the same general type.

16. Show that two terms of the form $a_1 \sin (kt + \epsilon_1)$ and $a_2 \sin (kt + \epsilon_2)$ combine into one term of the type mentioned in Ex. 15.

17. When a pendulum of length l swings through a small angle θ, its motion is very closely represented by the equation $\dfrac{d^2\theta}{dt^2} = -\dfrac{g}{l}\theta$, l being in feet, θ in radians, t in seconds. Show that $\theta = C_1 \sin kt + C_2 \cos kt$, where $k = \sqrt{g/l}$. Find C_1 and C_2 if $\theta = \alpha$ and the angular speed $\omega = 0$ when $t = 0$; and find the time required for one full swing.

18. A needle is suspended in a horizontal position by a torsion filament. When the needle is turned through a small angle from its position of equilibrium, the torsional restoring force produces an angular acceleration proportional to the angular displacement. Neglecting resistances, what will be the nature of the motion.

92. Damped vibrations. The curve

$$(1) \qquad y = e^{-t}$$

has been drawn in several examples; its relative rate of increase, $dy/dt \div y$, is -1. Hence the relative rate of increase of y is -1; or, the relative rate of decrease of y is $+1$.

If a vibration would follow the law

(2) $$y = a \sin kt$$

when not affected by friction, the formula

(3) $$y = ae^{-t} \sin kt,$$

in which a is replaced by ae^{-t}, expresses a corresponding **damped vibration,** in which the total amount of displacement y is equal at any

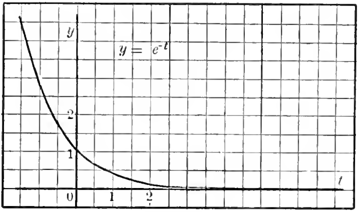


FIG. 40 *a*

instant t to the value given by a formula like (2) in which a *diminishes*, the relative rate of decrease in a being $+1$. The curve is shown in Fig. 40 (c); it may be obtained by drawing the ordinates in (1) multiplied by the corresponding ordinates in (2).

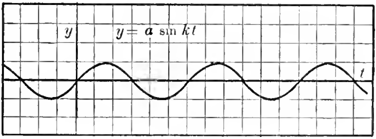


FIG. 40 *b*

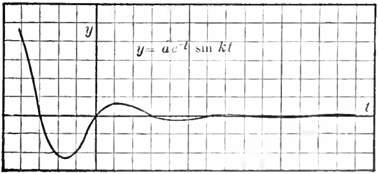


FIG. 40 *c*

Likewise,

(4) $$y = ae^{-bt}\sin(kt+\epsilon)$$

is a **damped vibration**, which may be written

(5) $$y = A\sin(kt+\epsilon),\quad \text{where } A = ae^{-bt}.$$

Here A is a *variable* decreasing amplitude, whose relative rate of decrease is $-dA/dx \div A = b$; that is, *the relative rate of decrease of A is constant.**

The successive derivatives of y, by (4), are:

$$\frac{dy}{dt} = ae^{-bt}[-b\sin(kt+\epsilon) + k\cos(kt+\epsilon)],$$

$$\frac{d^2y}{dt^2} = ae^{-bt}[(b^2-k^2)\sin(kt+\epsilon) - 2bk\cos(kt+\epsilon)],$$

whence it follows that

(6) $$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + (b^2+k^2)y = 0.\dagger$$

Equations which contain derivatives are called **differential equations**; thus (6) is the fundamental differential equation for damped vibrations.

EXERCISES XXXV. — DAMPED VIBRATIONS

1. Each of the following equations represents a damped harmonic vibration; find the speed and the acceleration in each case; and write an equation connecting the acceleration, the speed, and the value of y. Draw the graph of each equation.

(*a*) $y = e^{-t}\sin 2t$.

(*b*) $y = e^{-2t}\cos 4t$.

(*c*) $y = 5e^{-3t}\sin 7t$.

(*d*) $y = 2e^{-10t}\cos 5t$.

(*e*) $y = 2e^{-5t}\sin(2t + \pi/3)$.

(*f*) $y = 4e^{-.1t}\cos(3t - 5\pi/12)$.

* In common language, this is often expressed by saying that "the vibration dies away regularly," or "fades out uniformly." The fact that the relative rate of decrease of A is constant is the fundamental assumption.

† The equation (6) is often obtained directly and solved to obtain (4) as in Chapter X; the assumptions made in this work are equivalent to the assumption just mentioned, — that the relative rate of decrease of A is constant; this assumption is really the fundamental one, and its reasonableness is the real justification of the assumptions made when (6) is obtained first. The term in dy/dt, or v, *proportional to the velocity*, occurs only when "damping" (or friction) is considered. A similar equation governs electric vibrations.

2. The factor e^{-t^2} produces a more rapid damping effect. Draw $y = e^{-t^2} \sin t$, and compare it with $y = e^{-t} \sin t$. Find the speed and the acceleration in each case.

3. The factor $(1 + t^2)^{-1}$ (Example 2, p. 165) produces an effect similar to that of the factor e^{-t^2}. Draw $y = (1 + t^2)^{-1} \sin t$; find the speed and the acceleration.

4. Show that $y = e^{-t^2} \sin t$ satisfies the equation

$$d^2y/dt^2 + 4\,t\,(dy/dt) + (3 + 4\,t^2)\,y = 0.$$

5. Draw the curve $y = \operatorname{sech} t \cdot \sin t$; compare it with $y = e^{-t^2} \sin t$; find the speed and the acceleration.

93. Inverse Trigonometric Functions. Since the equations *

(1) $$y = \sin x, \quad x = \sin^{-1} y \quad (= \text{arc} \sin y)$$

are equivalent, it follows that

[XVI] $$\frac{d \sin^{-1} y}{dy} = \frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - y^2}},$$

a formula which may be written in other letters when convenient. It is evident that the radical should have the same sign as $\cos x$, *i.e.* + when x is in the 1st or 4th quadrants; − when x is in the 2d or 3d quadrants.

Likewise, from

(2) $$y = \cos x, \text{ or } x = \cos^{-1} y \quad (= \text{arc} \cos y).$$

[XVII] $$\frac{d \cos^{-1} y}{dy} = \frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{-\sin x} = \frac{-1}{\sqrt{1 - y^2}},$$

where the sign − applies when $\sin x$ is positive, *i.e.* for values of x in the 1st and 2d quadrants.

In like manner, the student may show that

[XVIII] $$\frac{d \tan^{-1} y}{dy} = \frac{1}{1 + y^2} \qquad \text{(all values of } y\text{)};$$

[XIX] $$\frac{d \operatorname{ctn}^{-1} y}{dy} = \frac{-1}{1 + y^2} \qquad \text{(all values of } y\text{)};$$

* The symbols $\sin^{-1} y$ and arc sin y will both be used to denote the *angle whose sine is y*. In writing such formulas as those on this page, the notation $\sin^{-1}$ is the shorter.

[XX] $$\frac{d \sec^{-1} y}{dy} = \frac{1}{y\sqrt{y^2-1}} \quad (x = \sec^{-1} y \text{ in 1st or 3d quadrant});$$

[XXI] $$\frac{d \csc^{-1} y}{dy} = \frac{-1}{y\sqrt{y^2-1}} \quad (x = \csc^{-1} y \text{ in 1st or 3d quadrant});$$

[XXII] $$\frac{d \operatorname{vers}^{-1} y}{dy} = \frac{1}{\sqrt{2y-y^2}} \quad (x = \operatorname{vers}^{-1} y \text{ in 1st or 2d quadrant});$$

94. Integrals of Irrational Functions. The preceding formulas are of little value as direct differentiation formulas. The reversed differentiations derived from them are very important, because they show how to obtain the integrals of certain simple irrational functions.

Thus, using the letter x in place of y, the formulas [XVI], [XVIII], [XX], and [XXII] become

[XVI]$_i$ $$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C, \text{ since } \frac{d \sin^{-1} x}{dx} = \frac{1}{\sqrt{1-x^2}},$$

[XVIII]$_i$ $$\int \frac{dx}{1+x^2} = \tan^{-1} x + C, \text{ since } \frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2},$$

[XX]$_i$ $$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C, \text{ since } \frac{d \sec^{-1} x}{dx} = \frac{1}{x\sqrt{x^2-1}},$$

[XXII]$_i$ $$\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{vers}^{-1} x + C, \text{ since } \frac{d \operatorname{vers}^{-1} x}{dx} = \frac{1}{\sqrt{2x-x^2}},$$

where C in each case denotes an arbitrary constant. Since $\sin^{-1} x + \cos^{-1} x = \pi/2$, the student may show that [XVII] leads to the same result as [XVI].

95. Illustrative Examples. A few illustrative examples will be given here; in Chapter VII many other integrals of irrational functions are found by means of those just written.

Example 1. Given $y = \sin^{-1}(x^2)$, to find dy/dx.

Method 1. Set $x^2 = u$. Then $dy/dx = (dy/du)(du/dx)$. Since $dy/du = d\sin^{-1}u/du = 1/\sqrt{1-u^2}$ and $du/dx = d(x^2)/dx = 2x$,

$$dy/dx = (1/\sqrt{1-u^2})\,2x = 2x/\sqrt{1-x^4}.$$

Method 2. $dy = d\sin^{-1}(x^2) = [1/\sqrt{1-(x^2)^2}]d(x^2) = (2x/\sqrt{1-x^4})dx.$

Notice that the resulting integral formula may be written

$$\int\frac{du}{\sqrt{1-u^2}} = \int\frac{2x\,dx}{\sqrt{1-x^4}} = \sin^{-1}u + C = \sin^{-1}(x^2) + C.$$

Example 2. To find the area under the curve $y = 1/(1+x^2)$ from the point where $x = 0$ to the point where $x = 1$.

Since $A = \int y\,dx$, we have

$$A\Big]_{x=0}^{x=1} = \int_{x=0}^{x=1}\frac{1}{1+x^2}\,dx = \tan^{-1}x\Big]_{x=0}^{x=1} = \pi/4 - 0 = \pi/4.$$

The fact that we are using radian measure for angles appears very prominently here. Draw the curve (by first drawing $y = 1 + x^2$) on a large scale on millimeter paper and actually count the small squares as a check on this result.

EXERCISES XXXVI.—INVERSE TRIGONOMETRIC FUNCTIONS

1. Differentiate each of the following functions:

(*a*) $\sin^{-1}x^3$.	(*e*) $\sin^{-1}(1-x^2)$.	(*i*) $\log\cos^{-1}x$.
(*b*) $\cos^{-1}(1+x)$.	(*f*) $x\cos^{-1}x$.	(*j*) $\sin^{-1}(xe^x)$.
(*c*) $\sin^{-1}(1/x)$.	(*g*) $\tan^{-1}(1/x^2)$.	(*k*) $x^2\tan^{-1}2\sqrt{x}$.
(*d*) $\tan^{-1}(2x)$.	(*h*) $e^x\sin^{-1}x$.	(*l*) $\sec^{-1}(x^2+4x)$.

2. Given $\operatorname{vers} x = 1 - \cos x$, show that the derivative of $\operatorname{vers}^{-1}x$ is $1/\sqrt{2x - x^2}$.

3. Given $\operatorname{exsec} x = \sec x - 1$, find the derivative of $\operatorname{exsec} x$.

4. Integrate the following functions; in case limits are stated, evaluate the integral:

(*a*) $\displaystyle\int_0^{\sqrt{3}}\frac{dx}{1+x^2}$.

(*b*) $\displaystyle\int_{1/2}^{1}\frac{dt}{\sqrt{1-t^2}}$.

(*c*) $\displaystyle\int_{-1}^{+1}\frac{dx}{1+x^2}$.

(*d*) $\displaystyle\int_1^2\frac{d\theta}{\theta\sqrt{\theta^2-1}}$.

(*e*) $\displaystyle\int\frac{dx}{1+4x^2}$. [Set $u = 2x$.]

(*f*) $\displaystyle\int\frac{dx}{\sqrt{1-4x^2}}$. [Set $u = 2x$.]

5. Find the areas between the x-axis and each of the following curves, between the limits stated :

(*a*) $y^2 = 1 + x^2y^2$; $x = 0$ to $x = 1/2$; $x = -1/2$ to $x = +1/2$.
(*b*) $y + x^2y = 1$; $x = 0$ to $x = 1$; $x = 0$ to $x = a$.
(*c*) $y^2 = 1 + 4x^2y^2$; $x = 0$ to $x = 1/4$; $x = -1/4$ to $x = +1/4$.
(*d*) $4x^2y + y + 1 = 0$; $x = 1$ to $x = 2$; $x = -1$ to $x = +1$.

6. Integrate after making the change of letters $u = 1 - x$:

(*a*) $\displaystyle\int \frac{dx}{\sqrt{1-(1-x)^2}}$. (*b*) $\displaystyle\int \frac{dx}{1+(1-x)^2}$. (*c*) $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}}$.

7. Show by differentiation that $\sin^{-1} x$ and $-\cos^{-1} x$ differ by a constant. Find the value of that constant by elementary trigonometry.

8. Show that $\sin^{-1} x$ and $\tan^{-1}[x/(1-x^2)^{1/2}]$ differ at most by a constant. By trigonometry, show that the two functions are equal.

9. Show that $\sin^{-1}(1-x)$ and $\text{vers}^{-1} x$ differ by a constant. Show that $\cos^{-1}(1-x) = \text{vers}^{-1} x$.

10. Show that the derivative of $\tan^{-1}[(e^x - e^{-x})/2]$ is $2/(e^x + e^{-x})$.

[NOTE. The function $\tan^{-1}[(e^x - e^{-x})/2]$, or $\tan^{-1}(\sinh x)$, is called the **Gudermannian** of x and is denoted by ***gd x*** : $\text{gd}\, x = \tan^{-1}(\sinh x)$. It follows from this exercise that $d\,\text{gd}\, x/dx = \text{sech}\, x$.]

11. From the fact that $d(\sinh x) = \cosh x\, dx$, show that the derivative of the **inverse hyperbolic sine** ($x = \sinh^{-1} u$ if $u = \sinh x$) is given by the equation $d(\sinh^{-1} u) = [1/(1+u^2)^{1/2}]\, du$. [See Ex. 4, p. 140.]

12. Show that $d\cosh^{-1} u = \pm[1/(u^2-1)^{1/2}]\, du$.

13. Show that $d\tanh^{-1} u = [1/(1-u^2)]\, du$.

96. Polar Coordinates. Equations of curves in polar coördinates frequently involve trigonometric functions. Given a curve C whose equation in polar coördinates is

$$\rho = f(\theta). \tag{1}$$

If PB is the arc of the circle about O with radius $\rho = OP$, then $BQ = \Delta\rho$; while arc $PB = \rho \cdot \angle POB = \rho\Delta\theta$. Hence

$$\frac{\Delta\rho}{\Delta\theta} = \rho\frac{BQ}{\text{arc } PB} = \rho \cdot \frac{BQ}{AQ} \cdot \frac{PA}{\text{arc } PB} \cdot \frac{AQ}{PA}. \tag{2}$$

Since $PA = \rho \sin \Delta\theta$, $PA/\text{arc } PB = \sin \Delta\theta/\Delta\theta$ approaches 1 by § 13, p. 19. Since $OA = \rho \cos \Delta\theta$,

$$\frac{AQ}{BQ} = \frac{\rho + \Delta\rho - \rho \cos \Delta\theta}{\Delta\rho} = 1 - \rho\, \frac{\cos \Delta\theta - 1}{\Delta\theta} \cdot \frac{\Delta\theta}{\Delta\rho},$$

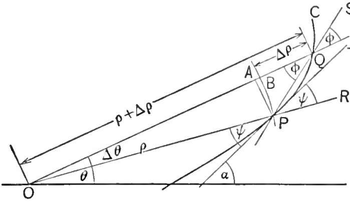


FIG. 41

which approaches 1 since

$$\lim_{\Delta\theta \doteq 0} \frac{\cos \Delta\theta - 1}{\Delta\theta} = \lim_{\Delta\theta \doteq 0} \frac{\cos (0 + \Delta\theta) - \cos 0}{\Delta\theta}$$

$$= \frac{d \cos \theta}{d\theta}\bigg]_{\theta=0} = - \sin \theta \bigg]_{\theta=0} = 0.$$

It follows that

$$(3) \qquad \frac{d\rho}{d\theta} = \lim_{\Delta\theta \doteq 0} \frac{\Delta\rho}{\Delta\theta} = \rho \lim_{\Delta\theta \doteq 0} \frac{AQ}{PA} = \rho \lim_{\Delta\theta \doteq 0} \operatorname{ctn} \phi = \rho \operatorname{ctn} \psi,$$

where $\phi = \angle DQS$ between the secant (S) and the radius vector OD and where $\psi = \angle RPT$ between the tangent (T) and the radius vector OR. The geometrical meaning of the derivative $d\rho/d\theta$ in polar coördinates is $\rho \operatorname{ctn} \psi$.

Dividing both sides of this equation by ρ:

$$\frac{d \log \rho}{d\theta} = \frac{d\rho}{d\theta} \div \rho = \operatorname{ctn} \psi = \tan \chi,$$

where $\chi\ (= 90° - \psi)$ is the angle between the curve and circle about O through the point P: *the relative rate of increase r_r of ρ with respect to θ is the tangent of the angle χ.* (See § 83, p. 147.)

The polar coördinate diagrams can therefore be used very effectively to represent quantities whose relative rates of change are important. For example, curves showing the growth of population of cities or countries may well be drawn on polar coördinate paper, the time being represented by the angle θ.

The angle α between the tangent (T) and the x-axis can be found after ψ has been found by means of the relation $\alpha = \theta + \psi$.

Example 1. Given the curve $\rho = e^{\theta}$, find $d\rho/d\theta$ and $r_r = \tan\chi = d\rho/d\theta \div \rho = d(\log \rho)/d\theta$. (See *Tables*, III, M.)

Since $\rho = e^{\theta}$, $d\rho/d\theta = e^{\theta}$, and $r = d\rho/d\theta \div \rho = 1$. Hence $\chi = \tan^{-1} 1 = \pi/4 = 45°$; this curve cuts every radius vector at the fixed angle 45°.

Physical experiments (see § 123,) in which it is suspected that the quantities measured follow a *compound interest law* can be tested by plotting in polar coördinates; the angle ψ (and therefore also χ) should be constant (see Ex. 4 below).

Example 2. Given $\rho = \sin 2\,\theta$, find ψ at the point where $\theta = \pi/8$.

$$\operatorname{ctn}\psi = \frac{d\rho}{d\theta} \div \rho = 2\cos 2\,\theta \div \sin 2\,\theta = 2\operatorname{ctn} 2\,\theta.$$

Hence $\operatorname{ctn}\psi = 2$ when $\theta = \pi/8$, whence $\psi = 26°\ 34'$.

EXERCISES XXXVII. — POLAR COÖRDINATES

1. Plot each of the following curves in polar coördinates; find the value of $\operatorname{ctn}\psi$ in general, and the value of ψ in degrees when $\theta = 0$, $\pi/6$, $\pi/4$, $\pi/2$, π.

(*a*) $\rho = 4\sin\theta$.	(*f*) $\rho = \theta$.	(*k*) $\rho = \sin 2\,\theta$.
(*b*) $\rho = 6\cos\theta - 5$.	(*g*) $\rho = \theta^2$.	(*l*) $\rho = 2\cos 3\,\theta$.
(*c*) $\rho = 3 + 4\cos\theta$.	(*h*) $\rho = 1/\theta$.	(*m*) $\rho = 3\sin(3\,\theta + 2\,\pi/3)$.
(*d*) $\rho = \tan\theta$.	(*i*) $\rho = e^{2\theta}$.	(*n*) $\rho = 3\cos\theta + 4\sin\theta$.
(*e*) $\rho = 2 + \tan^2\theta$.	(*j*) $\rho = e^{-4\theta}$.	(*o*) $\rho = 2/(1 - \cos\theta)$.

2. Show that $\operatorname{ctn}\psi$ is constant for the curve $\rho = ke^{a\theta}$.

3. Show that $\operatorname{ctn}\psi$ for the spiral $\rho = k\theta$ is greater than $\operatorname{ctn}\psi$ for $\rho = e^{\theta}$ when $\theta < 1$. Hence show that the former winds up more rapidly than the latter, as $\rho \doteq 0$.

4. Show that if the curves $\rho = e^{a\theta}$ are supposed drawn, for various values of a, any function $\rho = f(\theta)$ whose relative rate of change (loga-

rithmic derivative) is variable crosses them; show that the new curve moves across the others away from the origin if its relative rate of change is increasing as θ increases.

5. Find ctn ψ for each of the following curves:

(a) $\rho = p/(1 - e\cos\theta)$ (conic).
(b) $\rho = a\sec\theta \pm b$ (conchoid).
(c) $\rho = a(1 + \cos\theta)$ (cardioid).
(d) $\rho^2 = 2a^2\cos 2\theta$ (lemniscate).

6. Find the value of tan α [Fig. 41] in terms of the angles θ and ψ. Find tan α for each of the curves of Ex. 1, at the points specified.

97. Curvature. An important application of these formulas consists in finding a more accurate measure of the bending of a curve.

The *flexion* (§ 45, p. 71),

$$b = \frac{dm}{dx} = \frac{d^2y}{dx^2}, \tag{1}$$

is a crude measure of the bending; but it evidently depends upon the choice of axes, and changes when the axes are rotated, for example.

If we consider the rate of change of the inclination $\alpha = \tan^{-1} m$ with respect to the length of arc s, that is,

$$\lim_{\Delta s \doteq 0} \frac{\Delta\alpha}{\Delta s} = \frac{d\alpha}{ds}, \tag{2}$$

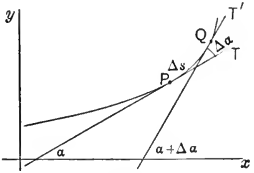


FIG. 42

it is evident that we have a measure of bending which does not depend on the choice of axes, since $\Delta\alpha$ and Δs are the same, even though the axes are moved about arbitrarily, or, indeed, before any axes are drawn. The quantity $d\alpha/ds$ is called the **curvature** of the curve at the point P, and is denoted by the letter K: *the curvature is the instantaneous rate of change of α per unit length of arc.*

Since $\alpha = \tan^{-1} m$, and since $ds^2 = dx^2 + dy^2$ (§ 62, p. 107), we have,

$$d\alpha = d\tan^{-1} m = \frac{1}{1+m^2}dm, \quad ds = \sqrt{1+m^2}\,dx,$$

where $m = dy/dx$; hence the curvature K is

$$(3) \qquad K = \frac{d\alpha}{ds} = \frac{\dfrac{1}{1+m^2}dm}{\sqrt{1+m^2}\,dx} = \frac{\dfrac{dm}{dx}}{(1+m^2)^{3/2}} = \frac{b}{(1+m^2)^{3/2}},$$

where $b = d^2y/dx^2$ (= flexion), and $m = dy/dx$ (= slope). It appears therefore that the flexion b when multiplied by the corrective factor $1/(1+m^2)^{3/2}$ gives a better measure of the bending, since K is independent of the choice of axes.

The reciprocal of K grows larger as the curve becomes flatter; it is called **the radius of curvature,** and is denoted by the letter R:

$$(4) \qquad R = \frac{1}{K} = \frac{ds}{d\alpha} = \frac{(1+m^2)^{3/2}}{b}.$$

It should be noticed that this concept agrees with the elementary concept of radius in the case of a circle, since $\Delta s = r\,\Delta\alpha$ in any circle of radius r.

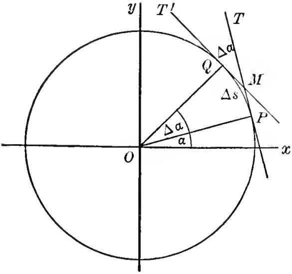


Fig. 43

Substituting the values of b and m, formulas (3) and (4) may be written in the forms

$$(5) \qquad K = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}};$$

$$(6) \qquad R = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}.$$

It is preferable, however, to calculate m and b first, and then substitute these values in (3) and (4).

Since $\sqrt{1+m^2} = \sec\alpha$ the formulas may also be written in the form $K = 1/R = b\cos^3\alpha$.

It is usual to consider only the **numerical values** of K, that is $|K|$, without regard to sign. Since K and b have the same sign, the value of K given by (3) will be negative when b is negative, *i.e.* when the curve is concave downwards (§ 46, p. 75). The same remarks apply to R, since $R = 1/K$.

EXERCISES XXXVIII. — CURVATURE

1. Calculate the curvature K and the radius of curvature $R = 1/K$ for each of the following curves:

(*a*) $y = x^2$. *Ans.* $R = (1 + 4x^2)^{3/2}/2$.

(*b*) $y = x^3$. *Ans.* $R = |(1 + 9x^4)^{3/2}/6x|$.

(*c*) $y = ax^2 + bx + c$. *Ans.* $R = |[1 + (2ax + b)^2]^{3/2}/2a|$.

(*d*) $y^2 = 4ax$. *Ans.* $R = (y^2 + 4a^2)^{3/2}/4a^2$.

(*e*) $xy = a^2$. *Ans.* $R = (x^2 + y^2)^{3/2}/2a^2$.

(*f*) $y = 3b^2x^2 - 2x^4$.

(*g*) $y = 6b^2x^2 - bx^3 + x^4$.

(*h*) $y = \sin x$.

(*i*) $y = \cos x - (\cos 2x)/2$.

(*j*) $y = e^x$.

(*k*) $y = (e^x + e^{-x})/2 = \cosh x$.

(*l*) $\dfrac{x^2}{a^2} \pm \dfrac{y^2}{b^2} = 1$. *Ans.* $R = a^2b^2\left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}\right)^{3/2}$.

(*m*) $\sqrt{x} + \sqrt{y} = \sqrt{a}$. *Ans.* $R = (x + y)^{3/2}/(2\sqrt{a})$.

(*n*) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $R = 3\sqrt[3]{axy}$.

(*o*) $y = a(e^{x/a} + e^{-x/a})/2$. *Ans.* $R = y^2/|a|$.

2. The *center of curvature* Q of a curve, corresponding to a point P, is obtained by drawing the normal at P and laying off R on this normal toward the concave side of the curve. Show that the coördinates of Q are

$$\alpha = x - R \sin \phi\,; \quad \beta = y + R \cos \phi,$$

where $\phi = \tan^{-1} dy/dx = \tan^{-1} m$. Show that these equations may also be written in the form:

$$\alpha = x - m\frac{1+m^2}{b},$$

$$\beta = y + \frac{1+m^2}{b},$$

where $m = dy/dx$, $b = d^2y/dx^2$. [See also § 155.]

3. Assuming the formulas of Ex. 2, find Q in terms of P for each of the curves in Ex. 1.

4. Plot the curve $y = x^2$; draw several of its normals, and lay off a distance equal to R on each of them toward the concave side of the curve.

The locus of these centers of curvature is called the **evolute** (see § 153). Draw the evolute.

[NOTE. The equation of the evolute may be found (§ 153) by eliminating x and y between the equations for α and β and the equation of the given curve.]

5. Plot the evolute, as in Ex. 4, for the curve 1(e) taking $a = 1$; for 1(h); for 1(k); for 1(l), taking $a = b = 1$.

6. Find the values of K, R, α, β for each of the following pairs of parameter equations:

(a) $x = a\cos t$, $y = a\sin t$. *Ans.* $R = |a|$.

(b) $x = a\cos^3 t$, $y = a\sin^3 t$. *Ans.* $R = 3|a\ (\sin 2t)/2|$.

(c) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. *Ans.* $R = 4|a\sin(\theta/2)|$.

(d) $x = t^2$, $y = t - t^2/3$. *Ans.* $R = (1 + t^2)^2/2$.

7. Show that the radius of curvature R of a curve whose equation is given in polar coördinates is

$$R = \frac{[\rho^2 + (d\rho/d\theta)^2]^{3/2}}{|\rho^2 + 2(d\rho/d\theta)^2 - \rho(d^2\rho/d\theta^2)|}.$$

[HINT. Since $R = ds/d\alpha$, and $\alpha = \theta + \psi$, § 96, p. 167, we have $d\alpha = d\theta + d\psi$; since $\psi = \tan^{-1}[\rho/(d\rho/d\theta)]$, $d\psi/d\theta$ can be found. Again, since $x = \rho\cos\theta$, $y = \rho\sin\theta$, $ds^2 = dx^2 + dy^2 = d\rho^2 + \rho^2\,d\theta^2$. Combine these to find $ds/d\alpha$.]

8. Assuming the formula of Ex. 7, find the radius of curvature of each of the following curves:

(a) $\rho = a^\theta$.	(d) $\rho = \cos\theta$.	(g) $\rho = a(1 + \cos\theta)$.
(b) $\rho = e^\theta$.	(e) $\rho = \sin 3\theta$.	(h) $\rho = 2/(1 + \cos\theta)$.
(c) $\rho\theta = a$.	(f) $\rho = a\sec 2\theta$.	(i) $\rho = a + b\cos\theta$.

98. Collection of Formulas. For convenience of reference, and for use in the exercises which follow, we collect here the formulas proved in this chapter. For convenience in printing they are given in differential notation. The formulas in ordinary type can all be obtained readily from those in black-faced type.

DIFFERENTIALS OF TRANSCENDENTAL FUNCTIONS

[VIII] $d\log_B x = \frac{dx}{x}\cdot\frac{M}{\log_{10} B} = \frac{dx}{x}\cdot\log_B e$, $[M = \log_{10} e = 0.4343]$.

[VIIIa] $\boldsymbol{d\log_e x = \frac{dx}{x}}$.

[IX] $dB^x = B^x \log_e B\,dx$.

[IXa] $\boldsymbol{de^x = e^x\,dx}$.

[X] $\boldsymbol{d\sin x = \cos x\,dx}$.

[XI] $\boldsymbol{d\cos x = -\sin x\,dx}$.

[XII] $\boldsymbol{d\tan x = \sec^2 x\,dx}$.

[XIII] $d\,\mathrm{ctn}\,x = -\csc^2 x\,dx$.

[XIV] $d\sec x = \sec x\tan x\,dx$.

[XV] $d\csc x = -\csc x\,\mathrm{ctn}\,x\,dx$.

[XVI] $\boldsymbol{d\sin^{-1} x = \frac{dx}{\sqrt{1-x^2}}}$, ($\sin^{-1}x$ in 1st or 4th quadrant).

[XVII] $d\cos^{-1} x = \frac{-dx}{\sqrt{1-x^2}}$, ($\cos^{-1}x$ in 1st or 2d quadrant).

[XVIII] $\boldsymbol{d\tan^{-1} x = \frac{dx}{1+x^2}}$, (all values of x).

[XIX] $d\,\mathrm{ctn}^{-1} x = \frac{-dx}{1+x^2}$, (all values of x).

[XX] $\boldsymbol{d\sec^{-1} x = \frac{dx}{x\sqrt{x^2-1}}}$, ($\sec^{-1}x$ in 1st or 3d quadrant).

[XXI] $d\csc^{-1} x = \frac{-dx}{x\sqrt{x^2-1}}$, ($\csc^{-1}x$ in 1st or 3d quadrant).

[XXII] $d\,\mathrm{vers}^{-1} x = \frac{dx}{\sqrt{2x-x^2}}$, ($\mathrm{vers}^{-1}x$ in 1st or 2d quadrant).

Other rules: See also **algebraic forms** (p. 40 or p. 52), **hyperbolic function forms** (Ex. 4, p. 140, Ex. 8, p. 141, and Exs. 11–13, p. 166); **Gudermannian** (Ex. 10, p. 166).

Integral formulas: See Chapter VII, and *Tables*, IV, A–H.

CHAPTER VII

TECHNIQUE — TABLES — SUCCESSIVE INTEGRATION

PART I. TECHNIQUE OF INTEGRATION

99. Question of Technique. Collection of Formulas. The discovery of indefinite integrals as reversed differentials was treated briefly, for certain algebraic functions, in Chapter V. We proceed to show how to integrate a variety of functions, but the majority are referred to **tables of integrals,** since no list can be exhaustive. See *Tables,* IV, A–H.

To every differential formula (pp. 52, 173) there corresponds a formula of integration:

$$\text{if } d\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})\,\boldsymbol{dx} \text{ then } \int \boldsymbol{f}(\boldsymbol{x})\,\boldsymbol{dx} = \boldsymbol{\phi}(\boldsymbol{x}) + \boldsymbol{C}.$$

The numbers assigned to the following formulas correspond to the number of the differential formula from which they come. The most important ones are set in black-faced type, except that black-faced type is not used when the formula is easy to remember intuitively. Certain omitted numbers correspond to relatively unimportant formulas.

FUNDAMENTAL INTEGRALS

[I]$_i$ $\quad$ *If* $\dfrac{\boldsymbol{dy}}{\boldsymbol{dx}} = \boldsymbol{0}$, *then* $\boldsymbol{y} =$ **constant.** [See § 58, p. 99.]

[The arbitrary constant C in each of the other rules results from this rule.]

[II]$_i$ $\quad$ $\int k\,f(x)\,dx = k\int f(x)\,dx + C.$

[III]$_i$ $\quad$ $\int \{f(x) + \phi(x)\}\,dx = \int f(x)\,dx + \int \phi(x)\,dx + C.$

[IV]$_i$ $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$, *when* $n \neq -1$. (See VIII.)

[VI]$_i$ $uv = \int d(uv) = \int u\,dv + \int v\,du + C$. ["Parts"]

[The corresponding formula [V]$_i$ for quotients is seldom used. See § 103.]

[VII]$_i$ $\left.\int f(u)\,du\right]_{u=\phi(x)} = \int f[\phi(x)]\,d\,\phi(x) + C$

[Substitution] $= \int f[\phi(x)]\,\dfrac{d\,\phi(x)}{dx}\,dx + C.$

[VIII]$_i$ $\int \dfrac{dx}{x} = \log x + C.$ [IX]$_i$ $\int e^x\,dx = e^x + C,$

[X]$_i$ $\int \cos x\,dx = \sin x + C.$ [XI]$_i$ $\int \sin x\,dx = -\cos x + C.$

[XII]$_i$ $\int \sec^2 x\,dx = \tan x + C.$

[XIII]$_i$ $\int \csc^2 x\,dx = -\operatorname{ctn} x + C.$

[XIV]$_i$ $\int \sec x \tan x\,dx = \sec x + C.$

[XV]$_i$ $\int \csc x \operatorname{ctn} x\,dx = -\csc x + C.$

[XVI]$_i$ $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C = -\cos^{-1} x + C'.$ [XVII]$_i$

[XVIII]$_i$ $\int \dfrac{dx}{1+x^2} = \tan^{-1} x + C = -\operatorname{ctn}^{-1} x + C'.$ [XIX]$_i$

[XX]$_i$ $\int \dfrac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C = -\csc^{-1} x + C'.$ [XXI]$_i$

[XXII]$_i$ $\int \dfrac{dx}{\sqrt{2x-x^2}} = \operatorname{vers}^{-1} x + C = -\operatorname{covers}^{-1} x + C'.$

The remaining differential formulas referred to on p. 173 give rise to other integral formulas; these will be found in the short **Table of Integrals**, *Tables*, IV, A–H.

100. Polynomials. Other Simple Forms. The rules [II], [III], [IV] are evidently sufficient without further explanation to integrate any polynomials and indeed many simple radical expressions. This work has been practiced in Chapter V extensively.

Attention is called especially to the fact that the rules [II] and [III] show that integration of a sum is in general simpler than integration of a product or a quotient. If it is possible, a product or a quotient should be replaced by a sum unless the integration can be performed easily otherwise. Thus the integrand $(1+x^2)/x$ should be written $1/x+x$; $(1+x^2)^2$ should be written $1+2x^2+x^4$; and so on. This principle appears frequently in what follows.

101. Substitution. Use of [VII]. As we have already done in simple cases in Chapters V and VI, substitution of a new letter may be used extensively, based on Rule [VII].

Example 1. To find $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}}$.

Set $u=x/a$, then $du=dx/a$, or $dx=a\,du$, and

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{a\,du}{\sqrt{a^2-a^2u^2}}=\int\frac{du}{\sqrt{1-u^2}}=\sin^{-1}u+C=\sin^{-1}\frac{x}{a}+C.$$

Check. $\displaystyle d\sin^{-1}\frac{x}{a}=\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\,d\left(\frac{x}{a}\right)=\frac{dx/a}{\sqrt{1-\frac{x^2}{a^2}}}=\frac{dx}{\sqrt{a^2-x^2}}$.

Example 2. To find $\displaystyle\int\sin 2x\,dx$.

Method 1. *Direct Substitution.*

$$\int\sin 2x\,dx=\tfrac{1}{2}\int\sin(2x)\,d(2x)=-\tfrac{1}{2}[\cos 2x+C]=-\tfrac{1}{2}\cos 2x+C'.$$

Check. $d(-\frac{1}{2}\cos 2x)=-\frac{1}{2}d\cos(2x)=+\frac{1}{2}\sin(2x)\,d(2x)=\sin 2x\,dx$.

Method 2. *Trigonometric Transformation and Substitution.*

$$\int\sin 2x\,dx=\int 2\sin x\cos x\,dx=-2\int\cos x\,d(\cos x)$$
$$=-(\cos x)^2+K=-\cos^2 x+K.$$

Notice that $\cos^2 x + K = 1/2 \cos 2x + C'$ since $\cos 2x = 2\cos^2 x - 1$. Do not be discouraged if an answer obtained seems different from an answer given in some table or book; two apparently quite different answers both may be correct, as in this example, for they may differ only by some constant.*

Whenever a prominent part of an integral is *accompanied by its derivative* as a coefficient of dx, *there is a strong indication of a* **desirable substitution**; thus if $\sin x$ occurs prominently and is accompanied by $\cos x\,dx$ substitute $u = \sin x$; if $\log x$ is prominent and is accompanied by $(1/x)dx$, set $u = \log x$; if any function $f(x)$ occurs prominently and is accompanied by $df(x)$, set $u = f(x)$. This is further illustrated in exercises below.

102. Substitutions in Definite Integrals. In evaluating definite integrals, the new letter introduced by a substitution may either be replaced by the original one after integration, or the values of the new letter which correspond to the given limits of integration may be substituted directly without returning to the original letter.

Example 1. Compute $\int_{x=0}^{x=\pi/2} \sin x \cos x\,dx$.

Method 1.

$$\int_{x=0}^{x=\pi/2} \sin x \cos x\,dx = \int_{x=0}^{x=\pi/2} u\,du\Big]_{u=\sin x} = \frac{u^2}{2}\Big]_{x=0}^{x=\pi/2} = \frac{\sin^2 x}{2}\Big]_{x=0}^{x=\pi/2} = \frac{1}{2}.$$

Method 2.

$$\int_{x=0}^{x=\pi/2} \sin x \cos x\,dx = \int_{u=0}^{u=1} u\,du\Big]_{u=\sin x} = \frac{u^2}{2}\Big]_{u=0}^{u=1} = \frac{1}{2},$$

since $u (= \sin x) = 0$ when $x = 0$, and $u = 1$ when $x = \pi/2$.

Care must be exercised to avoid errors when double-valued functions occur. The best precaution is to sketch a figure showing the relation between the old letter and the new one. In case there seems to be any doubt, it is safer to return to the original letter.

* Occasionally it is really difficult to show that two answers do actually differ by a constant in any other way than to show that the work in each case is correct and then appeal to the fundamental theorem (§ 58, p. 99).

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EXERCISES XXXIX. — ELEMENTARY INTEGRATION SUBSTITUTION

1. Integrate the following expressions:

(a) $\int (1+x)(1+x^2)\,dx.$

(b) $\int \frac{1+2x+3x^2}{x^2}\,dx.$

(c) $\int (a+bx)^2\,dx.$

(d) $\int \frac{(3x-2)^2}{x}\,dx.$

(e) $\int (e^x+e^{-x})^2\,dx.$

(f) $\int \frac{x^{3/2}-4x^{2/3}}{x^{1/2}}\,dx.$

(g) $\int (1+2x)^2\sqrt{x}\,dx.$

(h) $\int (x^2-2)(x^{1/2}+x^{2/3})\,dx.$

2. In the following integrals, carry out the indicated substitution; in answers, the arbitrary constant is here omitted for convenience in printing.

(a) $\int \sqrt{2x+3}\,dx$; set $u=2x+3$. *Ans.* $u^{3/2}/3=(2x+3)^{3/2}/3$.

(b) $\int \frac{dx}{2x+3}=\frac{1}{2}\log(2x+3)=\log\sqrt{2x+3}.$

(c) $\int \frac{dx}{1+(2x+3)^2}=\frac{1}{2}\tan^{-1}(2x+3).$

(d) $\int x\sqrt{1+x^2}\,dx$; set $u=1+x^2$. *Ans.* $u^{3/2}/3=(1+x^2)^{3/2}/3.$

(e) $\int \frac{x\,dx}{1+x^2}=\frac{1}{2}\log(1+x^2)=\log\sqrt{1+x^2}.$

(f) $\int \sin x\sqrt{\cos x}\,dx$; set $u=\cos x$. *Ans.* $-2u^{3/2}/3=-2(\cos^{3/2}x)/3.$

(g) $\int \cos x\sqrt{\sin x}\,dx=2(\sin^{3/2}x)/3.$

(h) $\int e^{1+x^2}x\,dx=\frac{1}{2}e^{1+x^2}.$

(i) $\int \cos x(1+2\sin x+3\sin^2 x)\,dx=\sin x+\sin^2 x+\sin^3 x.$

(j) $\int \sin^3 x\,dx=\int \sin x(1-\cos^2 x)dx=-\cos x+(\cos^3 x)/3.$

(k) $\int \cos(2x+3)\sin(2x+3)\,dx=\frac{1}{4}\sin^2(2x+3).$

(l) $\int \sin(1-3x)\cos^{3/2}(1-3x)\,dx=\frac{2}{15}\cos^{5/2}(1-3x).$

(m) $\int \frac{dx}{a^2+x^2}$; set $u=\frac{x}{a}$. *Ans.* $\frac{1}{a}\tan^{-1}u=\frac{1}{a}\tan^{-1}\frac{x}{a}.$

3. In the following integrals, find a substitution by inspection and complete the integration:

(a) $\int \frac{dx}{3x+4} = \frac{1}{3}\log(3x+4) = \log(3x+4)^{1/3}$.

(b) $\int \sqrt{1-2x}\,dx = -(1-2x)^{3/2}/3$.

(c) $\int \sin(2x-3)\,dx = -\frac{1}{2}\cos(2x-3)$.

(d) $\int \frac{x\,dx}{2x^2-5} = \log(2x^2-5)^{1/4}$.

(e) $\int \cos^3 x\,dx = \sin x - (\sin^3 x)/3$.

(f) $\int \cos x \sin^3 x\,dx = (\sin^4 x)/4$. (g) $\int \frac{\log x}{x}\,dx = \frac{1}{2}(\log x)^2$.

(h) $\int 2x\cos(1+x^2)\,dx = \sin(1+x^2)$.

(i) $\int \tan x \sec^2 x\,dx = (\tan^2 x)/2$.

(j) $\int (e^{2x}+e^{-2x})\,dx = \int e^{2x}\,dx + \int e^{-2x}\,dx = (e^{2x}-e^{-2x})/2$.

(k) $\int \cos^2 x\,dx = \int [(1+\cos 2x)/2]\,dx = x/2 + (\sin 2x)/4$.

(l) $\int \sin^2 x\,dx = \int [(1-\cos 2x)/2]\,dx = x/2 - (\sin 2x)/4$.

(m) $\int \cos^5 x\,dx = \sin x - 2(\sin^3 x)/3 + (\sin^5 x)/5$.

(n) $\int \operatorname{ctn} x\,dx = \int (\cos x/\sin x)\,dx = \log \sin x$.

(o) $\int \tan x\,dx = -\log\cos x = \log\sec x$.

(p) $\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right)$. (q) $\int \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$.

(r) $\int \frac{dx}{6+3x^2} = \frac{1}{3}\int \frac{dx}{2+x^2} = \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$.

(s) $\int \frac{dx}{\sqrt{12-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{3-x^2}} = \frac{1}{2}\sin^{-1}\left(\frac{x}{\sqrt{3}}\right)$.

4. Compute the values of the following definite integrals:

$$(a)\quad \int_{x=0}^{x=1} \frac{dx}{3+x^2} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}}\Bigg]_{x=0}^{x=1} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6\sqrt{3}}.$$

$$(b)\quad \int_{x=0}^{x=1} \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\Bigg]_{x=0}^{x=1} = \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

$$(c)\quad \int_{x=0}^{x=1} \frac{x\,dx}{3+x^2} = \tfrac{1}{2}\log_e(3+x^2)\Bigg]_{x=0}^{x=1} = \tfrac{1}{2}(\log_e 4 - \log_e 3) = .1438.$$

$$(d)\quad \int_{x=0}^{x=1} \frac{x\,dx}{\sqrt{2-x^2}} = -\sqrt{2-x^2}\Bigg]_{x=0}^{x=1} = -(\sqrt{1}-\sqrt{2}) = .4142.$$

$$(e)\quad \int_{x=0}^{x=\pi/3} \sin^3 x\,dx = \Big[-\cos x + (\cos^3 x)/3\Big]_{x=0}^{x=\pi/3} = 5/24.$$

$$(f)\quad \int_{x=0}^{x=1} x^2 e^{x^3}\,dx = e^{x^3}/3\Bigg]_{x=0}^{x=1} = e/3 - 1/3 = .5728.$$

$(g)\quad \displaystyle\int_{x=1}^{x=2} \sqrt{1+x}\,dx.$ $\qquad$ $(k)\quad \displaystyle\int_{x=0}^{x=1} \frac{dx}{6+2x^2}.$

$(h)\quad \displaystyle\int_{x=1}^{x=2} e^{-3x}\,dx.$ $\qquad$ $(l)\quad \displaystyle\int_{x=1}^{x=2} \frac{1+2x}{x+x^2}\,dx.$

$(i)\quad \displaystyle\int_{x=\pi/2}^{x=\pi/3} \sin^3 x \cos x\,dx.$ $\qquad$ $(m)\quad \displaystyle\int_{x=0}^{x=\pi/4} \cos^2 x\,dx.$

$(j)\quad \displaystyle\int_{x=0}^{x=\pi/6} \sin 2x\,dx.$ $\qquad$ $(n)\quad \displaystyle\int_{x=0}^{x=\pi/4} \cos^3 x\,dx.$

5. Find the area under the witch $y = 1/(a+bx^2)$ for $a = 9,\ b = 1$, from $x = 0$ to $x = 1$; for $a = 8,\ b = 2$, from $x = 1$ to $x = 10$.

6. Find the volume of the solid of revolution formed by revolving one arch of the curve $y = \sin x$ about the x-axis.

7. Find the area under the general catenary

$$y = a\cosh(x/a) = a(e^{x/a}+e^{-x/a})/2$$

from $x = 0$ to $x = a$.

8. Find the area of one arch of the cycloid

$$x = a(\theta - \sin\theta),\ y = a(1-\cos\theta).$$

9. Find the volume of the solid of revolution formed by revolving one arch of the cycloid about the x-axis.

10. Compare the area of one arch of the curve $y = \sin x$ with that of one arch of the curve $y = \sin 2x$; with that of one arch of $y = \sin^2 x$.

11. Show how any odd power of $\sin x$ or of $\cos x$ can be integrated by the device used in Ex. 2, (j).

12. Show how any power of $\sin x$ multiplied by an odd power of $\cos x$ can be integrated.

103. Integration by Parts. Use of Rule [VI]. — One of the most useful formulas in the reduction of an integral to a known form is [VI], which we here rewrite in the form

[VI′] $$\int u\,dv = uv - \int v\,du$$

called the formula for **integration by parts**. Its use is illustrated sufficiently by the following examples:

Example 1. $\int x \sin x\,dx$. Put $u = x$, $dv = \sin x\,dx$; then $du = dx$ and $v = \int \sin x\,dx = -\cos x$; hence,

$$\int x \sin x\,dx = -x \cos x + \int \cos x\,dx = -x\cos x + \sin x; \ (\textit{check}).$$

Example 2. $\int \log x\,dx$. Put $\log x = u$, $dx = dv$; then $du = (1/x)dx$, $v = x$, and

$$\int \log x\,dx = x \log x - \int x \cdot \frac{1}{x} \cdot dx = x \log x - \int dx = x \log x - x + C; \ (\textit{check}).$$

Example 3. $\int \sqrt{a^2 - x^2}\,dx$. Put $u = \sqrt{a^2 - x^2}$, $dv = dx$; then $v = x$, and

$$du = \frac{-x}{\sqrt{a^2 - x^2}}\,dx, \text{ and } \int \sqrt{a^2 - x^2}\,dx = x\sqrt{a^2 - x^2} + \int \frac{x^2\,dx}{\sqrt{a^2 - x^2}};$$

but, by Algebra, $$\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\int \sqrt{a^2 - x^2}\,dx + \int \frac{a^2 dx}{\sqrt{a^2 - x^2}};$$

hence

$$2\int \sqrt{a^2 - x^2}\,dx = x\sqrt{a^2 - x^2} + \int \frac{a^2\,dx}{\sqrt{a^2 - x^2}}$$
$$= x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + C.$$

This important integral gives, for example, the area of the circle $x^2 + y^2 = a^2$, since one fourth of that area is

$$\int_{x=0}^{x=a} \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]_{x=0}^{x=a} = \frac{1}{2}\left[a^2 \cdot \frac{\pi}{2}\right] = \frac{\pi a^2}{4}.$$

EXERCISES XL. — INTEGRATION BY PARTS

1. Carry out each of the following integrations:

(*a*) $\int x \cos x \, dx = x \sin x + \cos x + C.$

(*b*) $\int xe^x \, dx = e^x(x - 1) + C.$ [Hint. $u = x$, $dv = e^x \, dx$.]

(*c*) $\int x \log x \, dx = -x^2/4 + (x^2 \log x)/2 + C.$

(*d*) $\int x^2 \log x \, dx = -x^3/9 + (x^3 \log x)/3 + C.$

(*e*) $\int x^2 e^x \, dx = e^x(x^2 - 2x + 2) + C.$ [Hint. Use [VI] twice.]

(*f*) $\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1 - x^2} + C.$ [Hint. $u = \sin^{-1} x$.]

(*g*) $\int \tan^{-1} x \, dx = x \tan^{-1} x - \log(1 + x^2)^{1/2} + C.$

(*h*) $\int x^2 \tan^{-1} x \, dx = (x^3 \tan^{-1} x)/3 - x^2/6 + \log(1 + x^2)^{1/6} + C.$

(*i*) $\int x(e^x - e^{-x})/2 \, dx = \int x \sinh x \, dx = x \cosh x - \sinh x + C.$

(*j*) $\int x^2 e^{2x} \, dx = e^{2x}(x^2/2 - x/2 + 1/4) + C.$

(*k*) $\int e^x \sin x \, dx = e^x(\sin x - \cos x)/2.$ [Set $u = e^x$; use [VI] twice.]

(*l*) $\int e^{3x} \cos 2x \, dx = e^{3x}(2 \sin 2x + 3 \cos 2x)/13.$

(*m*) $\int e^{-x} \sin 4x \, dx = -e^{-x}(4 \cos 4x + \sin 4x)/17.$

(*n*) $\int e^{ax} \cos nx \, dx = e^{ax}(n \sin nx + a \cos nx)/(a^2 + n^2).$

2. Show that $\int P(x) \tan^{-1} x \, dx$, where $P(x)$ is any polynomial, reduces to an algebraic integral by means of [VI]. Show how to integrate the remaining integral.

3. Show that $\int P(x)\log x\,dx$, where $P(x)$ is any polynomial, reduces to an algebraic integral by means of [VI]. Show how to integrate the remaining integral.

4. Express $\int x^n e^{ax}\,dx$ in terms of $\int x^{n-1}e^{ax}\,dx$. Hence show that $\int P(x)e^{ax}\,dx$ can be integrated, where $P(x)$ is any polynomial.

5. Carry out each of the following integrations:

(a) $\int (1+2x-x^2)\log x\,dx$. (c) $\int (x^2-2x+3)e^{-2x}\,dx$.

(b) $\int (3x^2+4x-1)\tan^{-1}x\,dx$. (d) $\int (x^3-5)e^{4x}\,dx$.

6. From the rule for the derivative of a quotient, derive the formula $\int (1/v)du = u/v + \int (u/v^2)dv$. Show that this rule is equivalent to [VI] if u and v in [VI] are replaced by $1/v$ and u, respectively.

7. Integrate $\int e^x \sin x\,dx$ by applying [VI] once with $u = e^x$, then with $u = \sin x$, and adding.

8. Integrate $\int e^{ax}\sin nx\,dx$ by the scheme of Ex. 7.

9. Find the values of each of the following definite integrals:

(a) $\int_{x=1}^{x=e}\log x\,dx$. (d) $\int_{x=0}^{x=1}(1+3x^2)\tan^{-1}x\,dx$.

(b) $\int_{x=0}^{x=1} xe^{-x}\,dx$. (e) $\int_{x=0}^{x=\pi/2} e^{-2x}\cos 3x\,dx$.

(c) $\int_{x=0}^{x=1/2}\sin^{-1}x\,dx$. (f) $\int_{x=1}^{x=2}(e^x-e^{-x})\,dx$.

10. Find the area of one arch of the curve $y = e^{-x}\sin x$.

11. Find the area beneath each of the following curves: (a) $y = xe^{-x}$, (b) $y = x^2e^{-x}$, (c) $y = x^3e^{-x}$, from $x = 0$ to $x = 1$.

12. Compare the area beneath the curve $y = \log x$ from $x = 1$ to $x = e$ with the approximate result obtained by using the adaptation of the prismoid formula, Ex. 6, p. 128.

13. Show that the sum of the area beneath the curve $y = \sin x$ from $x = 0$ to $x = k$ and that beneath the curve $y = \sin^{-1}x$ from $x = 0$ to $x = \sin k$ is the area of a rectangle whose diagonal joins $(0, 0)$ and $(k, \sin k)$.

104. Rational Functions. To integrate any polynomial, rules [II], [III], [IV] are sufficient. When the integrand is rational, but *fractional,* the formulas [VIII] and [XVIII] are required.

Example 1.

$$\int\left(2x-\frac{1}{x}+\frac{2}{1+x^2}\right)dx = x^2 - \log\, x + 2 \text{ arc } \tan x + C.$$

Moreover, it may be necessary to prepare the expression for integration.

Example 2. $\dfrac{N}{D}=\dfrac{2x^4+x^2+2x-1}{x^3+x}=2x-\dfrac{1}{x}+\dfrac{2}{1+x^2}.$

The method of preparation is as follows: Any rational fraction N/D in which the degree of N is greater than or equal to the degree of D may be replaced by an integral quotient Q together with a proper fraction R/D, where R is of lower degree than D. This results from ordinary algebraic division, where Q is the quotient and R is the remainder. Thus,

$$\frac{N}{D}=\frac{2x^4+x^2+2x-1}{x^3+x}=2x-\frac{x^2-2x+1}{x^3+x}.$$

The rational fraction R/D can then be broken up, by algebra, into a sum of **proper partial fractions**, whose denominators are the real factors of the first or second degrees of D, and integral powers of these factors. *If D has only simple factors, there are just as many proper partial fractions as there are factors, each partial fraction taking as its denominator one factor.*

Thus in Example 2, the factors of D are x and x^2+1; hence we write

$$-\frac{R}{D}=\frac{x^2-2x+1}{x^3+x}=\frac{A}{x}+\frac{Bx+C}{1+x^2}$$

where A, B, C are at present unknown constants.*

* Notice that the numerators inserted are just one less in degree than the corresponding denominators: this is because it is known that the resulting partial fractions will be **proper** fractions. The numerator should be written in the most general form for which the fraction is a proper fraction.

Clearing of fractions, we find

$$x^2 - 2x + 1 = A + Ax^2 + Bx^2 + Cx;$$

comparing the coefficients of like terms,

$$A + B = 1,\ C = -2,\ A = 1, \text{ whence } A = 1,\ B = 0,\ C = -2$$

and $$-\frac{R}{D} = \frac{x^2 - 2x + 1}{x^3 + x} = \frac{1}{x} + \frac{-2}{1 + x^2}, \quad \text{(Check by addition)}$$

whence the example can be completed as above.

If D consists of one factor raised to an integral power, and if the degree of R is not less than the degree of that factor, then R/D can be simplified still further by ordinary division, using the factor to the first power as a divisor.

Example 3. Given $\dfrac{R}{D} = \dfrac{3x + 2}{(x + 1)^2}$, to find $\displaystyle\int \frac{R}{D}\,dx$.

Dividing R by $(x + 1)$, we find

$$\frac{R}{x + 1} = 3 + \frac{-1}{x + 1} = 3 - \frac{1}{x + 1};$$

hence $$\frac{R}{D} = \frac{R}{x + 1} \cdot \frac{1}{x + 1} = \frac{3}{x + 1} - \frac{1}{(x + 1)^2}; \quad \textit{(check)}.$$

Therefore

$$\int \frac{R}{D}\,dx = \int \frac{3\,dx}{x + 1} - \int \frac{dx}{(x + 1)^2} = 3 \log (x + 1) + \frac{1}{x + 1} + C; \quad \textit{(check)}.$$

Example 4. Given $\dfrac{R}{D} = \dfrac{x^3 + 2x^2 + 2}{(x^2 + 1)^2}$, to find $\displaystyle\int \frac{R}{D}\,dx$.

Dividing R by $(x^2 + 1)$, we find

$$\frac{R}{x^2 + 1} = x + 2 - \frac{x}{x^2 + 1}, \quad \frac{R}{D} = \frac{x + 2}{(x^2 + 1)} - \frac{x}{(x^2 + 1)^2};$$

and $$\int \frac{R}{D}\,dx = \int \frac{x\,dx}{x^2 + 1} + \int \frac{2\,dx}{x^2 + 1} - \int \frac{x\,dx}{(x^2 + 1)^2}$$

$$= \frac{1}{2} \log (x^2 + 1) + 2 \tan^{-1} x + \frac{1}{2(x^2 + 1)}; \quad \textit{(check)}.$$

If, finally, D has several factors, some repeated and some simple, we proceed as before to make as many distinct *proper* partial fractions as there are distinct factors, using as denominators the simple factors as before, *and the repeated factors raised to the same power as that in which they occur in D.**

Example 5. Given $\dfrac{R}{D}=\dfrac{4x^4+8x^3+6x^2+6x+5}{(3x+2)(x^2+1)^2}$, to find $\displaystyle\int\frac{R}{D}dx.$

Set
$$\frac{R}{D}=\frac{a}{3x+2}+\frac{bx^3+cx^2+dx+e}{(x^2+1)^2},$$

clear of fractions, and equate the coefficients of like powers:

$$a+3b=4,\ 2b+3c=8,\ 2a+2c+3d=6,\ 2d+3e=6,\ a+2e=5;$$

whence
$$a=1,\ b=1,\ c=2,\ d=0,\ e=2;$$

and
$$\frac{R}{D}=\frac{1}{3x+2}+\frac{x^3+2x^2+2}{(x^2+1)^2}.$$

The integration is then completed as in Ex. 4.

EXERCISES XLI. — RATIONAL FRACTIONS

1. Carry out the following integrations:

(*a*) $\displaystyle\int\frac{dx}{2x+5}=\log\sqrt{2x+5}.$

(*b*) $\displaystyle\int\frac{dx}{x^2-1}=\frac{1}{2}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx=\frac{1}{2}\log\frac{x-1}{x+1}.$

(*c*) $\displaystyle\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\frac{x-a}{x+a}.$

(*d*) $\displaystyle\int\frac{dx}{x^2+2x+5}=\int\frac{dx}{(x+1)^2+4}=\frac{1}{2}\tan^{-1}\frac{x+1}{2}.$

(*e*) $\displaystyle\int\frac{dx}{x^2+4x+5}=\tan^{-1}(x+2).$

* This simple rule, together with the algebraic reduction by long division just mentioned, is perfectly general and is always successful whenever the denominator D can be factored.

(*f*) $\int \frac{dx}{4x^2 + 4x + 2} = [\tan^{-1}(2x+1)]/2.$

(*g*) $\int \frac{dx}{x^2 + 2x - 3} = \int \frac{dx}{(x+1)^2 - 4} = \frac{1}{4} \log \frac{x-1}{x+3}.$

(*h*) $\int \frac{dx}{4x^2 + 4x - 8} = \frac{1}{12} \log \frac{2x-2}{2x+4} = \frac{1}{12} \log \frac{x-1}{x+2}.$

2. In the following integrals, first prepare the integrand for integration as in § 104; then complete the integrations.

(*a*) $\int \frac{3x-7}{x^2 - 5x + 6} dx = \log(x-2) + 2\log(x-3) = \log[(x-2)(x-3)^2].$

(*b*) $\int \frac{4x+15}{5x+3x^2} dx = 3\log x - \frac{5}{3}\log(3x+5).$

(*c*) $\int \frac{x^2+2}{x^2+1} dx = x + \tan^{-1} x.$

(*d*) $\int \frac{x-1}{x^2+2x+2} dx = \log\sqrt{x^2+2x+2} - 2\tan^{-1}(1+x).$

(*e*) $\int \frac{dx}{x^4 + 7x^2 + 12} = \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - \frac{1}{2} \tan^{-1} \frac{x}{2}.$

(*f*) $\int \frac{7x+9}{9+9x-4x^2} dx.$

(*g*) $\int \frac{14x-25}{4x^2-25} dx.$

(*h*) $\int \frac{2x-13}{x^2+10x+25} dx.$

(*i*) $\int \frac{8x^3 - 36x^2 - 2}{(2x+1)(2x-5)} dx.$

(*j*) $\int \frac{2x^2-20}{(x^2-1)(x^2-4)} dx.$

(*k*) $\int \frac{x^2-45}{2x^3-18x} dx.$

(*l*) $\int \frac{2x+3}{x^3+x^2-2x} dx.$

(*m*) $\int \frac{x-1}{x^2+x+1} dx.$

(*n*) $\int \frac{dx}{x^3+1}.$

(*o*) $\int \frac{2x^2-x+3}{x+x^2+x^3} dx.$

(*p*) $\int \frac{dx}{x^3+x^2+x+1}.$

(*q*) $\int \frac{x^2\,dx}{x^4+x^2-2}.$

(*r*) $\int \frac{x^2+1}{x^2-x} dx.$

(*s*) $\int \frac{3x^2-4}{(x+1)^3} dx.$

(*t*) $\int \frac{3x-1}{x^2(x+1)^2} dx.$

3. Derive the following formulas:

(a) $\displaystyle\int \frac{dx}{(ax+b)(mx+n)} = \frac{-1}{an-bm} \log \frac{mx+n}{ax+b}.$

(b) $\displaystyle\int \frac{x\,dx}{(x+a)(x+b)} = \frac{1}{a-b} \log \frac{(x+a)^a}{(x+b)^b}.$

(c) $\displaystyle\int \frac{x\,dx}{(x^2+a)(x+b)} = \frac{b}{a+b^2} \log \frac{\sqrt{x^2+a}}{x+b} + \frac{\sqrt{a}}{a+b^2} \tan^{-1} \frac{x}{\sqrt{a}}.$

4. Derive each of the formulas Nos. 18–24, *Tables*, IV, A.

5. Evaluate each of the following definite integrals:

(a) $\displaystyle\int_{x=1}^{x=2} \frac{8-3x-x^2}{x(x+2)^2}\,dx.$ (b) $\displaystyle\int_{x=2}^{x=4} \frac{15-7x+3x^2-3x^3}{x^4+5x^3}\,dx.$

(c) $\displaystyle\int_{x=3/5}^{x=12/5} \frac{5x(5x-4)}{(5x-2)^3}\,dx.$

[NOTE. Further practice in definite integration may be had by inserting various limits in the previous exercises.]

6. Carry out each of the following integrations after reducing them to algebraic form by a proper substitution:

(a) $\displaystyle\int \frac{\sin x}{1+\cos^2 x}\,dx.$ (b) $\displaystyle\int \frac{\cos x}{4-\sin^2 x}\,dx.$ (c) $\displaystyle\int \frac{e^x\,dx}{1-e^{2x}}.$

(d) $\displaystyle\int \sec x\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx = \frac{1}{2} \log \frac{1+\sin x}{1-\sin x}.$

(e) $\displaystyle\int \csc x\,dx.$ (g) $\displaystyle\int \operatorname{csch} x\,dx.$ (j) $\displaystyle\int \frac{\sec^2 x}{\tan x - \tan^2 x}\,dx.$

(f) $\displaystyle\int \operatorname{sech} x\,dx.$ (i) $\displaystyle\int \frac{\sin x \cos x}{1+\cos^3 x}\,dx.$ (k) $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx.$

105. Rationalization of Linear Radicals. If the integrand is rational except for a radical of the form $\sqrt{ax+b}$, the substitution of a new letter for the radical,

$$r = \sqrt{ax+b},$$

renders the new integrand rational.

Example 1. Find $\int \frac{x+\sqrt{x+2}}{1+x}\,dx.$

Setting $r=\sqrt{x+2}$, we have $x=r^2-2$ and $dx=2\,r\,dr$; hence

$$\int \frac{x+\sqrt{x+2}}{1+x}\,dx=\int \frac{r^2+r-2}{r^2-1}\,2\,r\,dr=2\int \frac{r+2}{r+1}\,r\,dr$$

$$=2\int\left(r+1-\frac{1}{r+1}\right)dr=r^2+2\,r-2\log(r+1)+C$$

$$=x+2+2\sqrt{x+2}-2\log(\sqrt{x+2}+1)+C.$$

The same plan — *substitution of a new letter for the essential radical* — is successful in a large number of cases, including all those in which the radical is of one of the forms:

$$x^{1/n},\quad (ax+b)^{1/n},\quad \left(\frac{ax+b}{cx+d}\right)^{1/n},$$

where n is an integer. Integral powers of the essential radical may also occur in the integrand.

106. Quadratic Irrationals: $\sqrt{a+bx\pm x^2}$. If the integral involves a quadratic irrational, either of several methods may be successful, and at least one of the following always succeeds:

(*A*) If the quadratic $Q=a+bx\pm x^2$ can be factored into real factors, we have

$$\sqrt{Q}=\sqrt{(\alpha+x)(\beta\pm x)}=(\alpha+x)\sqrt{\frac{\beta\pm x}{\alpha+x}};$$

and the method of § 105 can be used. The resulting expressions are sometimes not so simple, however, as those found by one of the following processes.

(*B*) If the term in x^2 is positive, either of the substitutions

$$\sqrt{Q}=t+x,\quad \sqrt{Q}=t-x,$$

will be found advantageous. One of these substitutions may lead to simpler forms than the other in a given example.

(*C*) Completing the square under the radical sign throws the radical in the form

$$\sqrt{Q} = \sqrt{\pm k \pm (x \pm c)^2};$$

the substitution $x \pm c = y$ certainly simplifies the integral, and may throw it in a form which can be recognized instantly.

Example 1. Let $\sqrt{Q} = \sqrt{x^2 \pm a^2}$; show the effect of substituting $\sqrt{Q} = t - x$.

If $\sqrt{x^2 \pm a^2} = t - x$, we find

$$x = \frac{t^2 \mp a^2}{2t},\ dx = \frac{t^2 \pm a^2}{2t^2}\,dt,\ \sqrt{Q} = t - x = \frac{t^2 \pm a^2}{2t};$$

and the transformed integrand is surely rational. Carrying out these transformations in the simple examples which follow, we find

$$\text{(i)}\quad \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \int \left(\frac{t^2 \pm a^2}{2t^2} \div \frac{t^2 \pm a^2}{2t}\right) dt = \int \frac{dt}{t} = \log t + C$$

$$= \log(x + \sqrt{Q}) + C, \qquad \text{where} \qquad Q = x^2 \pm a^2.$$

$$\text{(ii)}\quad \int \sqrt{x^2 \pm a^2}\, dx = \int \frac{t^2 \pm a^2}{2t} \cdot \frac{t^2 \pm a^2}{2t^2}\, dt = \int \left(\frac{t}{4} \pm \frac{a^2}{2t} + \frac{a^4}{4t^3}\right) dt$$

$$= \frac{t^4 - a^4}{8t^2} \pm \frac{a^2}{2} \log t = \frac{x\sqrt{Q}}{2} \pm \frac{a^2}{2} \log(x + \sqrt{Q}) + C.$$

These integrals are important and are repeated in the Table of Integrals, *Tables*, IV, C, 33, 45*a*. Many other integrals can be reduced to these two or to that of Ex. 3, p. 181, or to Rules [XVI] or [XX] by process (*C*) above.

EXERCISES XLII. — INTEGRALS INVOLVING RADICALS

1. Verify the following integrations:

(a) $\int x\sqrt{1+x}\,dx = \frac{6x-4}{15}(1+x)^{3/2}.$

(b) $\int \frac{dx}{(x-1)\sqrt{2-x}} = \log\frac{\sqrt{2-x}-1}{\sqrt{2-x}+1}.$

(c) $\int \frac{dx}{(x-1)\sqrt{x-2}} = 2\tan^{-1}\sqrt{x-2}.$

(d) $\int \sqrt{\frac{1+x}{1-x}}\,dx = \sin^{-1}x - \sqrt{1-x^2}.$

(e) $\int \frac{dx}{(ax+2b)\sqrt{ax+b}} = \frac{2}{a\sqrt{b}}\tan^{-1}\sqrt{\frac{ax+b}{b}}.$

(f) $\int \frac{dx}{x^2\sqrt{x+1}} = -\frac{\sqrt{x+1}}{x} + \frac{1}{2}\log\frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}.$

(g) $\int (a+bx)^{3/2}dx = \frac{2}{5b}(a+bx)^{5/2}.$

(h) $\int \frac{dx}{(a+bx)^{3/2}} = -\frac{2}{b\sqrt{a+bx}}.$

2. Carry out the following integrations:

(a) $\int \frac{dx}{x^2\sqrt{x-1}}.$ (b) $\int \frac{x\,dx}{\sqrt{a-x}}.$ (c) $\int \frac{x\,dx}{\sqrt{ax+b}}.$

(d) $\int \frac{x^2\,dx}{\sqrt{x+1}}.$ (e) $\int \frac{x^3}{\sqrt{1-x}}\,dx.$ (f) $\int \frac{x\,dx}{(x+2)\sqrt{2x+4}}.$

(g) $\int \frac{x^{1/2}+x}{1-x^{1/3}}\,dx.$ (h) $\int \frac{x^{1/4}}{9+x^{1/2}}\,dx.$ (i) $\int \frac{x\,dx}{(3x-2)^{4/3}}.$

(j) $\int \frac{\sqrt{1+x}}{x}\,dx.$ (k) $\int \frac{\sqrt{ax-b}}{x}\,dx.$ (l) $\int \frac{\sqrt{x+1}}{x+5}\,dx.$

(m) $\int_{x=0}^{x=1/4} \frac{\sqrt{x}}{1-\sqrt{x}}\,dx.$ (n) $\int_{x=4}^{x=9} \frac{1+\sqrt{x}}{1-\sqrt{x}}\,dx.$ (o) $\int_{x=0}^{x=3} \frac{x\,dx}{1+\sqrt{1+x}}.$

(p) $\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}}.$ (q) $\int \frac{dx}{\sqrt[3]{x-8}+2}.$ (r) $\int \sqrt{\frac{x+3}{x+1}}\,dx.$

3. Carry out the following integrations by first making an appropriate substitution:

(a) $\displaystyle\int \frac{\cos x}{3+\sqrt{\sin x}}\,dx.$ (d) $\displaystyle\int \frac{\sin x\sqrt{\cos x}}{1-2\sqrt{\cos x}}\,dx.$

(b) $\displaystyle\int \frac{e^x\,dx}{1+e^{x/3}}.$ (e) $\displaystyle\int \frac{1+\sqrt{\sin x}}{1-\sqrt{\sin x}}\cos x\,dx.$

(c) $\displaystyle\int \frac{\sec^2 x\,dx}{\sqrt{2+3\tan x}}.$ (f) $\displaystyle\int \frac{\sin x\,dx}{(2-3\cos x)^{3/2}}.$

4. Substitution of a new letter for the essential radical is immediately successful in the following integrals:

(a) $\displaystyle\int x\sqrt{1+x^2}\,dx.$ (d) $\displaystyle\int \frac{(1+x)dx}{\sqrt{2+2x+x^2}}.$ (g) $\displaystyle\int x^2\sqrt{a+bx^3}\,dx.$

(b) $\displaystyle\int \frac{x^3\,dx}{\sqrt{1+x^2}}.$ (e) $\displaystyle\int \frac{x\,dx}{(a+bx^2)^{3/2}}.$ (h) $\displaystyle\int \frac{x^5}{\sqrt{a+bx^3}}\,dx.$

(c) $\displaystyle\int x(1+x^2)^{3/2}\,dx.$ (f) $\displaystyle\int \frac{x\,dx}{(a+bx^2)^{p/q}}.$ (i) $\displaystyle\int \frac{x^{n-1}\,dx}{\sqrt{a+bx^n}}.$

5. Carry out the following integrations:

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} = \log(x+\sqrt{x^2-1}).$

(b) $\displaystyle\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\log\frac{\sqrt{x^2+a^2}-a}{x}.$

(c) $\displaystyle\int \frac{dx}{(x+a)\sqrt{x^2-a^2}} = \frac{1}{a}\sqrt{\frac{x-a}{x+a}}.$

(d) $\displaystyle\int \frac{dx}{(1+2x^2)\sqrt{1+x^2}} = \tan^{-1}\frac{x}{\sqrt{x^2+1}}.$

(e) $\displaystyle\int \frac{x+1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x - \sqrt{1-x^2}.$

(f) $\displaystyle\int \frac{\sqrt{1-x^2}}{1+x^2}\,dx = \sqrt{2}\tan^{-1}\frac{x\sqrt{2}}{\sqrt{1-x^2}} - \sin^{-1}x.$

(g) $\displaystyle\int \frac{2\,dx}{\sqrt{4x^2-1}}.$ (i) $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}}.$ (k) $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx.$

(h) $\displaystyle\int \frac{dx}{(x-1)\sqrt{1-x^2}}.$ (j) $\displaystyle\int \frac{dx}{(1+x)\sqrt{1+x^2}}.$ (l) $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2+4}}.$

6. The following integrations may be performed by the methods of § 106; note especially method (C), which consists in completing the square under the radical.

(a) $\int \frac{dx}{\sqrt{x^2+x+1}} = \log(1 + 2x + 2\sqrt{x^2+x+1})$.

(b) $\int \frac{dx}{\sqrt{1+x-x^2}} = \sin^{-1}\frac{2x-1}{\sqrt{5}}$.

(c) $\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\frac{x-a}{a} = \text{vers}^{-1}x$ [+ const.].

(d) $\int \frac{dx}{x\sqrt{1+x+x^2}} = \log\frac{2+x-2\sqrt{1+x+x^2}}{x}$.

(e) $\int \frac{dx}{x\sqrt{3x^2+4x-4}} = \frac{1}{2}\sin^{-1}\frac{x-2}{2x}$.

(f) $\int \frac{dx}{\sqrt{2x^2+x+1}}$. (g) $\int \frac{dx}{\sqrt{1+x-2x^2}}$. (h) $\int \frac{dx}{\sqrt{1-2x-x^2}}$.

(i) $\int \frac{dx}{\sqrt{6x-x^2-5}}$.

(j) $\int \frac{dx}{x\sqrt{x^2+2x+3}}$.

(k) $\int \frac{dx}{(x+4)\sqrt{x^2+3x-4}}$.

(l) $\int \sqrt{1+x+x^2}\,dx$.

(m) $\int \sqrt{3x^2+10x+9}\,dx$.

(n) $\int x\sqrt{8+x-x^2}\,dx$.

7. Integrate by "Parts," [VI], the following integrals:

(a) $\int x\sin^{-1}x\,dx$.

(b) $\int \frac{\sin^{-1}x}{x^2}dx$.

(c) $\int x\cos^{-1}x\,dx$.

(d) $\int (3x-2)\sin^{-1}x\,dx$.

(e) $\int \frac{2+x^2}{x^2}\cos^{-1}x\,dx$.

(f) $\int (\sin^{-1}x + 2x\cos^{-1}x)dx$.

8. Show that $\int P(x)\sin^{-1}x\,dx$ reduces by means of [VI] to an integral whose integrand contains no other radical than $\sqrt{1-x^2}$, if $P(x)$ is any polynomial.

9. Show that by means of the substitution $x = \sin\theta$ the integrals $\int dx/\sqrt{1-x^2}$ and $\int d\theta$ are equivalent.

10. Reduce the following integrals to trigonometric integrals; then complete the integration:

(a) $\displaystyle\int \frac{(1+x)\,dx}{\sqrt{1-x^2}} = \int (1+\sin\theta)\,d\theta$, if $x = \sin\theta$.

Ans. $\theta - \cos\theta = \sin^{-1}x - \sqrt{1-x^2}$.

(b) $\displaystyle\int \frac{dx}{(1-x^2)\sqrt{1-x^2}} = \int \frac{d\theta}{\cos^2\theta}$, if $x = \sin\theta$. *Ans.* $\tan\theta = \dfrac{x}{\sqrt{1-x^2}}$.

(c) $\displaystyle\int x\sqrt{x^2-1}\,dx = \int \tan^2\theta\sec^2\theta\,d\theta$, if $x = \sec\theta$.

Ans. $(\tan^3\theta)/3 = (x^2-1)^{3/2}/3$.

(d) $\displaystyle\int \frac{x\,dx}{\sqrt{1+x^2}} = \int \sec\theta\tan\theta\,d\theta$, if $x = \tan\theta$. *Ans.* $\sec\theta = \sqrt{1+x^2}$.

11. Reduce the following integrals to algebraic integrals; then complete the integration:

(a) $\displaystyle\int \frac{d\theta}{1-\sin\theta} = \int \frac{dx}{(1-x)\sqrt{1-x^2}}$, if $x = \sin\theta$. [See 5 (*h*).]

(b) $\displaystyle\int \sec\theta\,d\theta = \int \frac{dx}{\sqrt{x^2-1}}$, if $x = \sec\theta$. [See 5(*a*).]

(c) $\displaystyle\int \frac{\sec\theta\,d\theta}{1+2\tan^2\theta} = \int \frac{dx}{(1+2x^2)\sqrt{1+x^2}}$, if $x = \tan\theta$. [See 5(*d*).]

(d) $\displaystyle\int \frac{d\theta}{1+\sin^2\theta}$. (e) $\displaystyle\int \frac{d\theta}{1+\tan\theta}$. (f) $\displaystyle\int \frac{\sec\theta\,d\theta}{1+\sec\theta}$.

107. Elliptic and Other Integrals. If the essential radical in the integrand is the square root of a cubic or of a polynomial of higher degree, or a cube root or higher root, the integrals are usually beyond the scope of this book.

If the only irrationality is $\sqrt{Q}$, where Q is a polynomial of the third or fourth degree, the integral is called an **elliptic integral.** While no treatment of these integrals is given here, they are treated briefly in tables of integrals, and their values have been computed in the form of tables.* See *Tables*, V, *D*, *E*.

108. Binomial Differentials. Among the forms which are shown in tables of integrals to be reducible to simpler ones are the so-called **binomial differentials**:

$$\int (ax^n + b)^p x^m \, dx.$$

It is shown by integration by parts that such forms can be replaced by any one of the following combinations, where u stands for $(ax^n + b)$:

(1) $$\int u^p x^m \, dx = (A_1)\, u^p x^{m+1} + (B_1) \int u^{p-1} x^m \, dx,$$

(2) $$\int u^p x^m \, dx = (A_2)\, u^{p+1} x^{m+1} + (B_2) \int u^{p+1} x^m \, dx,$$

(3) $$\int u^p x^m \, dx = (A_3)\, u^{p+1} x^{m+1} + (B_3) \int u^p x^{m+n} \, dx,$$

(4) $$\int u^p x^m \, dx = (A_4)\, u^{p+1} x^{m-n+1} + (B_4) \int u^p x^{m-n} \, dx,$$

where A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4 are certain constants.

These rules may be used either by direct substitution from

* Some idea of these quantities may be obtained by imagining some person ignorant of logarithms. Then $\int (1/x)dx$ would be beyond his powers, and we should tell him "*values of the integral* $\int (1/x)dx$ *can be found tabulated*," which is precisely what is done in tables of Napierian logarithms. Of course as little as possible is tabulated; other allied forms are reduced to those tabulated by means of special formulas, given in the tables. Tables of the values of integrals are often computed even though the integral can be found in terms of known functions: thus tables of values of $\log [x + \sqrt{x^2+1}] = \int dx/\sqrt{x^2 + 1^2}$ are to be found under the name *inverse hyperbolic sine of x* $(= \sinh^{-1} x)$; see p. 140, and *Tables*, V, C; and also II, II; IV, C, 33.

a table of integrals in which the values of the constants are given in general † (see *Tables,* IV, D, 51–54), or we may denote the unknown constants by letters and find their values by differentiating both sides and comparing coefficients.

Example 1. $\int \frac{dx}{(ax^2+b)^{3/2}} = A\frac{x}{(ax^2+b)^{1/2}} + B\int \frac{dx}{(ax^2+b)^{1/2}}$, by **(2)**

Differentiating and comparing coefficients of x^2 and x^0, we find $B = 0$ and $A = 1/b$; hence

$$\int \frac{dx}{(ax^2+b)^{3/2}} = \frac{x}{b\sqrt{(ax^2+b)}}; \text{ (check.)}$$

Example 2. $\int \frac{x^3\,dx}{(ax^2+b)^{3/2}} = \frac{Ax^2}{(ax^2+b)^{1/2}} + B\int \frac{x\,dx}{(ax^2+b)^{3/2}}$, by **(4)**.

Here $A = 1/a,\ B = -2\,b/a,$

and
$$\int \frac{x^3dx}{(ax^2+b)^{3/2}} = \frac{ax^2+2\,b}{a^2\sqrt{(ax^2+b)}},$$

109. General Remarks. The student will see that integration is largely a trial process, the success of which is dependent upon a ready knowledge of algebraic and trigonometric transformations. Skill will come only from constant practice. A very considerable help in this practice is a table of integrals (see *Tables,* IV, A–H. The student should apply his intelligence in the use of such tables, testing the results there given, endeavoring to see how they are obtained, studying the classification of the table; in brief, *mastering* the table, not becoming a slave to it.

In the list which follows, many examples can be done by the processes mentioned above. The exercises which are starred (*) may be reserved for practice in using a table of integrals.

† Such formulas are called **reduction formulas**; many other such formulas — notably for trigonometric functions — are given in tables of integrals. (See *Table* IV, E*a*, 57, 60, 64, etc.) It is strongly advised that no effort be made to memorize any of these forms, — not even the skeleton forms given above. A far more profitable effort is to grasp the essential notion of the types of changes which can be made in these and other integrals, so that good judgment is formed concerning the possibility of integrating given expressions. Then the actual integration is usually performed by means of a table. See also *Tables,* IV, E_a, 78, 82 (*b*); E_b, 85, 86; E_c, 92–94; E_d, 98, 106; B, 17 (*b*), 25; etc.

EXERCISES XLIII. — GENERAL INTEGRATION

[As stated above, the exercises which are starred (*) may be integrated by use of tables of integrals.]

1. (*a*) $\int \frac{x^3 - 4x^2 + 1}{(x-2)^2}\,dx.$ (*d*) $\int \frac{x^3 + 1}{x^4 - 3x^3 + 3x^2 - x}\,dx.$

(*b*) $\int \frac{3x^2 - 17x + 21}{(x-2)^3}\,dx.$ (*e*) $\int \frac{7x^2 + 7x - 176}{x^3 - 9x^2 + 6x + 56}\,dx.$

(*c*) $\int \frac{x^3 - 2x^2 + 7x + 4}{(x^2-1)^2}\,dx.$ (*f*) $\int \frac{x^2 - 3x + 3}{x^3 - 4x^2 - 7x + 10}\,dx.$

2. (*a*) $\int \frac{2x + 1}{(x^2+1)^2}\,dx.$ (*e*) $\int \frac{6x^5\,dx}{(5 - 7x^3)^3}.$ (*i*) $\int \frac{3\,dx}{5 - 7x + 2x^2}.$

(*b*) $\int \frac{x^2\,dx}{12 + 5x^2}.$ (*f*) $\int \frac{x\,dx}{4 - x^4}.$ (*j*) $\int \frac{x^2\,dx}{5 + 2x + x^2}.$

(*c*) $\int \frac{27x^6\,dx}{2 + 3x^2}.$ (*g*) $\int \frac{x^9\,dx}{2 + 5x^4}.$ (*k*) $\int \frac{dx}{x^2 + 4x + 2}.$

(*d*) $\int \frac{6x^5\,dx}{3 - 2x^3}.$ (*h*) $\int \frac{dx}{x(3 + 5x^6)}.$ (*l*) $\int \frac{x\,dx}{x^2 + 4x + 2}.$

3. (*a*) $\int x\sqrt{1 + 2x}\,dx.$ (*c*) $\int \frac{\sqrt{x}}{x - 1}\,dx.$ (*e*) $\int \frac{x^2\,dx}{\sqrt{4 + 2x}}.$

(*b*) $\int \frac{x^2}{\sqrt{3x + 5}}\,dx.$ (*d*) $\int \frac{dx}{x\sqrt{x - a}}.$ (*f*) $\int x^2\sqrt{5 + 2x}\,dx.$

4. (*a*) $\int x\sqrt[3]{x - 4}\,dx.$ (*c*) $\int \frac{dx}{\sqrt[n]{(a + bx)^p}}.$ (*e*) $\int x\sqrt[3]{3x + 7}\,dx.$

(*b*) $\int x\sqrt[3]{a + bx}\,dx.$ (*d*) $\int \frac{x^2\,dx}{3\sqrt[3]{x + 2}}.$ (*f*) $\int \frac{2 + x}{\sqrt{3 - x}}\,dx.$

(*g*) $\int \frac{(8 + 2x)^{2/3}}{3x}\,dx.$ (*i*) $\int \frac{x\,dx}{\sqrt{1 + x} + \sqrt[3]{1 + x}}.$

(*h*) $\int \frac{1 + \sqrt{x} - \sqrt[3]{x^2}}{1 + \sqrt[3]{x}}\,dx.$ (*j*) $\int \frac{dx}{\sqrt{1 + x} + \sqrt[4]{1 + x}}.$

(k) $\int \frac{\sqrt[4]{x}\,dx}{\sqrt[3]{x}+\sqrt{x}}$. (m) $\int \frac{x\,dx}{\sqrt[4]{a+bx}}$. (o) $\int x\sqrt[4]{3x+7}\,dx$.

(l) $\int \frac{x^2\,dx}{\sqrt[3]{1+x}}$. (n) $\int \frac{2+x}{\sqrt[3]{3-x}}\,dx$. (p) $\int x\sqrt[3]{a+bx^2}\,dx$.

(q) $\int x^3(a+x^2)^{1/3}\,dx$. (r) $\int x^5(1+x^3)^{1/3}\,dx$.

5.* (a) $\int \frac{dx}{(x^2+1)^2}$. (c) $\int \frac{3x+2}{(x^2+3)^3}\,dx$. (e) $\int \frac{dx}{(x^2+2x+5)^2}$.

(b) $\int \frac{dx}{(x^2+5)^3}$. (d) $\int \frac{5x-3}{(2x^2-1)^2}\,dx$. (f) $\int \frac{(2x+1)\,dx}{(x^2+6x+10)^3}$.

6.* (a) $\int \frac{dx}{x^3\sqrt{x^2-4}}$. (d) $\int (a^2-x^2)^{3/2}\,dx$. (g) $\int \frac{dx}{x(a+bx^3)^{2/3}}$.

(b) $\int \frac{x^2\,dx}{\sqrt{x^2-a^2}}$. (e) $\int (x^2-a^2)^{3/2}\,dx$. (h) $\int \frac{(1+x^2)^{2/5}}{x^3}\,dx$.

(c) $\int \frac{x^5\,dx}{(7+4x^3)^{2/3}}$. (f) $\int \frac{x^7\,dx}{(9x^4-3)^{3/4}}$. (i) $\int x^5(a+bx^3)^4\,dx$.

7.* (a) $\int \sin^4 x\,dx$. (b) $\int \tan^3 x\,dx$. (c) $\int \sin^2 x\cos^4 x\,dx$.

(d) $\int \frac{d\theta}{2+\sin\theta}$. (e) $\int \frac{d\theta}{2-3\cos\theta}$. (f) $\int \frac{d\theta}{2+5\sin\theta}$.

(g) $\int \cos^6 \alpha\,d\alpha$. (h) $\int \operatorname{ctn}^2 3x\,dx$. (i) $\int \sin^{3/2}\theta\cos^3\theta\,d\theta$.

8.* (a) $\int_0^1 \frac{2x+3}{(x-2)^2}\,dx$. (d) $\int_0^1 \frac{3x-2}{2x^2-3}\,dx$. (g) $\int_0^2 \frac{(1+x^2)\,dx}{\sqrt{4-x^2}}$.

(b) $\int_1^2 \frac{x^2+2}{(2x-1)^3}\,dx$. (e) $\int_2^5 \frac{dx}{x\sqrt{2x+3}}$. (h) $\int_1^2 \frac{dx}{\sqrt{x^2+1}}$.

(c) $\int_0^1 \frac{3x-2}{2x^2+3}\,dx$. (f) $\int_2^3 \frac{3x-2}{\sqrt{x^2-3}}\,dx$. (i) $\int_1^2 \frac{dx}{\sqrt{x^2-1}}$.

(j) $\int_0^2 \frac{dx}{(2x+1)(x^2+2)}$. (k) $\int_{-2}^{-1} \frac{dx}{\sqrt{x^2-8x}}$.

9.* Find the values of the following definite integrals by using the tabulated numerical values: *Tables*, V, A–H :

(*a*) $\int_{x=0}^{x=1.5} e^x\,dx.$ (*c*) $\int_{x=0}^{x=1.4} e^{-x}\,dx.$ (*e*) $\int_{x=3.5}^{x=4.1} \frac{1}{x-2}\,dx.$

(*b*) $\int_{x=1.2}^{x=2.5} e^{2x}\,dx.$ (*d*) $\int_{x=1}^{x=2.3} \frac{1}{x}\,dx.$ (*f*) $\int_{x=2.1}^{x=3.2} \frac{1}{x^2-1}\,dx.$

(*g*) $\int_{x=0}^{x=1.4} \frac{e^x+e^{-x}}{2}\,dx = \int_{x=0}^{x=1.4} \cosh x\,dx.$ (*h*) $\int_{x=0}^{x=2.3} \sinh x\,dx.$

(*i*) $\int_{x=1}^{x=2} \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x\Big]_{x=1}^{x=2}.$ (*j*) $\int_{x=0}^{x=1} \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x\Big]_{x=0}^{x=1}.$

(*k*) $\int_{x=2}^{x=3.6} \frac{dx}{\sqrt{x^2-4}} = \cosh^{-1}\frac{x}{2}\Big]_{x=2}^{x=3.6}.$

(*l*) $\int_{x=0}^{x=14.4} \frac{dx}{\sqrt{x^2+9}} = \sinh^{-1}\frac{x}{3}\Big]_{x=0}^{x=14.4}.$

(*m*) $\int_{\theta=0^\circ}^{\theta=30^\circ} \frac{d\theta}{\sqrt{1-(1/4)\sin^2\theta}} = F(\tfrac{1}{2}, 30^\circ),$ *Tables*, V, D.

(*n*) $\int_{\theta=0^\circ}^{\theta=45^\circ} \sqrt{1-(1/4)\sin^2\theta}\,d\theta = E(\tfrac{1}{2}, 45^\circ),$ *Tables*, V, E.

(*o*) $\int_{\theta=15^\circ}^{\theta=45^\circ} \frac{d\theta}{\sqrt{1-.04\sin^2\theta}}.$ (*p*) $\int_{\theta=30^\circ}^{\theta=90^\circ} \sqrt{1-.25\sin^2\theta}\,d\theta.$

(*q*) $\int_{x=0}^{x=1/2} \frac{dx}{\sqrt{1-x^2}\sqrt{1-.25x^2}} = \int_{\theta=0^\circ}^{\theta=30^\circ} \frac{d\theta}{\sqrt{1-.25\sin^2\theta}},$ if $x = \sin\theta.$

(*r*) $\int_{x=1/2}^{x=1} \sqrt{\frac{1-.36x^2}{1-x^2}}\,dx = \int_{\theta=30^\circ}^{\theta=90^\circ} \sqrt{1-.36\sin^2\theta}\,d\theta,$ if $x = \sin\theta.$

(*s*) $\int_{x=1/2}^{x=1} \frac{dx}{\sqrt{1-x^2}\sqrt{1-.49x^2}}.$ (*t*) $\int_{x=\sqrt{2}/2}^{x=\sqrt{3}/2} \sqrt{\frac{1-.16x^2}{1-x^2}}\,dx.$

[NOTE. Many of the exercises in Lists XXXIX–XLII may be used for additional practice in use of the tables.]

10.* Show that the even powers of $\sin x$ can be integrated by reduction to the integral $\int \sin^2 x\,dx$.

11. Show that odd powers of $\cos x$ can be integrated readily without a table.

12.* Show that any power of $\tan x$ can be integrated by reduction to $\int \tan x\,dx$ or to $\int \tan^2 x\,dx$.

13. Show that any even power of $\sec x$ can be integrated by splitting off one factor $\sec^2 x$ and then using the relation $\sec^2 x = 1 + \tan^2 x$.

14. Show that $\int x^n e^x\,dx$ can be integrated by the repeated use of Rule [VI].

Hence show that $\int P(x) e^x\,dx$ can be integrated, if $P(x)$ is any polynomial.

15. If $\int f(x)\,dx = \phi(x)$ show that $\int f(x) \tan^{-1} x\,dx$ can be reduced to $\int [\phi(x)/(1 + x^2)]\,dx$ by Rule [VI].

State a similar result for the integral $\int f(x) \sin^{-1} x\,dx$.

16. Show that the integrals which result from breaking up a rational fraction whose denominator has only simple linear factors can be expressed in terms of simple powers and logarithms.

17. Show that a simple quadratic factor in the denominator of a rational fraction gives rise to a term in the final answer which contains an arc tangent or a logarithm.

18. Show how to integrate terms of each of the following types, and show that no others arise in integrating rational fractions:

(*a*) $\displaystyle\int \frac{dx}{ax + b}$. (*d*) $\displaystyle\int \frac{Ax + B}{x^2 + a^2}\,dx$. (*g*) $\displaystyle\int \frac{Ax + B}{ax^2 + bx + c}\,dx$.

(*b*) $\displaystyle\int \frac{dx}{(ax + b)^2}$. (*e*)* $\displaystyle\int \frac{Ax + B}{(x^2 + a^2)^2}\,dx$. (*h*)* $\displaystyle\int \frac{Ax + B}{(ax^2 + bx + c)^2}\,dx$.

(*c*)* $\displaystyle\int \frac{dx}{(ax + b)^n}$. (*f*)* $\displaystyle\int \frac{Ax + B}{(x^2 + a^2)^n}\,dx$. (*i*)* $\displaystyle\int \frac{Ax + B}{(ax^2 + bx + c)^n}\,dx$.

PART II. IMPROPER AND MULTIPLE INTEGRALS

110. Limits Infinite. Horizontal Asymptote. If a curve approaches the x-axis as an asymptote, it is conceivable that the total area between the x-axis, the curve, and a left-hand vertical boundary may exist; by this total area we mean the *limit* of the area from the left-hand boundary out to any vertical line $x = m$, as m becomes infinite.

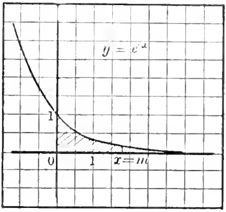


FIG. 44

Example 1. The area under the curve $y=e^{-x}$ from the y-axis to the ordinate $x = m$ is

$$A\Big]_{x=0}^{x=m} = \int_{x=0}^{x=m} e^{-x}\,dx = 1 - e^{-m}.$$

As m becomes infinite e^{-m} approaches zero; hence

$$A\Big]_{x=0}^{x=\infty} = \int_{x=0}^{x=\infty} e^{-x}\,dx = \lim_{m \doteq \infty} \int_{x=0}^{x=m} e^{-x}\,dx = \lim_{m \doteq \infty} (1 - e^{-m}) = 1,$$

and we say that the total area under the curve $y = e^{-x}$ from $x = 0$ to $x = +\infty$ is 1.

Example 2. The area under the hyperbola $y = 1/x$ from $x = 1$ to $x = m$ is

$$A\Big]_{x=1}^{x=m} = \int_{x=1}^{x=m} \frac{dx}{x} = \log x\Big]_{x=1}^{x=m} = \log m.$$

As m becomes infinite, $\log m$ becomes infinite, and

$$\lim_{m \doteq \infty} \left\{ A\Big]_{x=1}^{x=m} \right\} = \lim_{m \doteq \infty} \log m$$

does not exist; hence we say that the total area between the x-axis and the hyperbola from $x = 1$ to $x = \infty$ *does not exist.**

* This is the standard short expression to denote what is quite obvious, — that the area up to $x=m$ becomes infinite as m becomes infinite. This result makes any consideration of the area up to $x = \infty$ perfectly useless; hence the expression "fails to exist," which is slightly more general.

111. Integrand Infinite. Vertical Asymptotes. If the function to be integrated becomes infinite, the situation is precisely similar to that of § 110; graphically, the curve whose area is represented by the integral has in this case a vertical asymptote.

If $f(x)$ becomes infinite at one of the limits of integration, $x = b$, we define the integral, as in § 110, by a limit process:

$$\int_{x=a}^{x=b} f(x)\,dx = \lim_{c \doteq 0} \int_a^{b-c} f(x)\,dx.$$

A similar definition applies if $f(x)$ becomes infinite at the lower limit, as in the following example.

Example 1. The area between the curve $y = 1/\sqrt{x}$ and the two axes, from $x = 0$ to $x = 1$, is

$$A\Big]_{x=0}^{x=1} = \int_{x=0}^{x=1} \frac{1}{\sqrt{x}}\,dx = \lim_{c \doteq 0}\left[\int_{x=c}^{x=1} \frac{1}{\sqrt{x}}\,dx\right]$$

$$= \lim_{c \doteq 0}\left[2\sqrt{x}\right]_{x=c}^{x=1} = \lim_{c \doteq 0}\left[\;2 - 2\sqrt{c}\;\right] = 2.$$

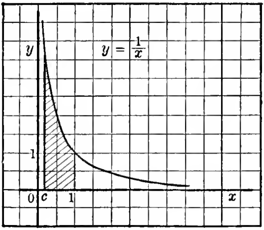


FIG. 45

Example 2. The area between the hyperbola $y = 1/x$, the vertical line $x = 1$, and the two axes, *does not exist.* For,

$$\int_{x=c}^{x=1} \frac{1}{x}\,dx = \log x\Big]_{x=c}^{x=1} = -\log c,$$

but lim $(-\log c)$ as $c \doteq 0$ does not exist, for $-\log c$ becomes infinite as $c \doteq 0$.

Example 3. The area between the curve $y = 1/\sqrt[3]{x-1}$, its asymptote $x = 1$, and the line $x = 2$ is

$$\int_1^2 \frac{dx}{\sqrt[3]{x-1}} = \lim_{c \doteq 0} \int_{1+c}^2 \frac{dx}{\sqrt[3]{x-1}} = \frac{3}{2} \lim_{c \doteq 0} (1 - c^{2/3}) = \frac{3}{2}.$$

112. Precautions. It is dangerous to apply limits of integration between which the integrand becomes infinite or is otherwise discontinuous.

Example 1. Show that $\int_{-1}^{+1} 1/x^2 dx$ does not exist. The ordinate $y = 1/x^2$ becomes infinite as x approaches zero, *i.e.* the y-axis is a vertical asymptote. Hence to find the given integral we must proceed as in § 111, breaking the original integral into two parts:

$$\mathrm{I} = \int_{x=c}^{x=1} \frac{1}{x^2}\, dx = -\frac{1}{x}\Big]_{x=c}^{x=1} = \frac{1}{c} - 1.$$

$$\mathrm{II} = \int_{x=-1}^{x=-c} \frac{1}{x^2}\, dx = -\frac{1}{x}\Big]_{x=-1}^{x=-c} = \frac{1}{c} - 1.$$

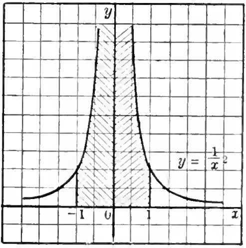


FIG. 46

The limit of neither exists since $1/c$ becomes infinite as $c \doteq 0$; hence the given integral does not exist.

Carelessness in such cases results in absurdly false answers; thus if no attention were paid to the nature of the curve, some person might write:

$$A\Big]_{x=-1}^{x=+1} = \int_{x=-1}^{x=+1} \frac{1}{x^2} dx = (\text{sic !}) \frac{-1}{x}\Big]_{x=-1}^{x=+1} = (\text{sic !}) - 1 - 1 = -2,$$

which is ridiculous (see Fig. 46).

The only general rule is to follow the principles of §§ 110–111 in all cases of infinite limits or discontinuous integrands. Such integrals are called **improper integrals**.

EXERCISES XLIV. — IMPROPER INTEGRALS

1. Verify the following results:

(*a*) $\displaystyle\int_{-1}^{+1}\frac{dx}{x^{2/3}} = 6.$

(*b*) $\displaystyle\int_{-1}^{+1}\frac{dx}{x^3}$ is non-existent.

(*c*) $\displaystyle\int_0^1\frac{dx}{x^n}$ is $\begin{cases}\text{determinate if } n < 1,\\ \text{non-existent if } n \geqq 1.\end{cases}$

(*d*) $\displaystyle\int_0^1\frac{dx}{\sqrt{1-x}} = 2.$

(*f*) $\displaystyle\int_{1.5}^2\frac{dx}{\sqrt{2x-3}} = 1.$

(*e*) $\displaystyle\int_0^a\frac{dx}{\sqrt{a-x}} = 2\sqrt{a}.$

(*g*) $\displaystyle\int_{1.5}^2\frac{dx}{2x-3}$ is non-existent.

(*h*) $\displaystyle\int_{1.5}^2\frac{dx}{(2x-3)^n}$ is $\begin{cases}\text{determinate if } n < 1,\\ \text{non-existent if } n \geqq 1.\end{cases}$

State a similar rule for $\displaystyle\int_a^b\frac{dx}{(hx+k)^n}$.

(*i*) $\displaystyle\int_0^a\frac{dx}{\sqrt{a^2-x^2}} = \frac{\pi}{2}.$

(*k*) $\displaystyle\int_0^a\frac{x^2dx}{\sqrt{ax-x^2}} = \frac{3\pi a^2}{8}.$

(*j*) $\displaystyle\int_0^1\frac{x\,dx}{\sqrt{1-x^2}} = 1.$

(*l*) $\displaystyle\int_1^{+1}\frac{dx}{x^2+5x+4}$ is non-existent.

2. Show that the integrals $\displaystyle\int_0^{\frac{\pi}{2}}\tan x\,dx,\ \int_0^{\frac{\pi}{2}}\cot x\,dx,\ \int_0^{\frac{\pi}{2}}\sec x\,dx,$ $\displaystyle\int_0^1\frac{\log x\,dx}{x}$ are all non-existent.

3. Verify each of the following results:

(*a*) $\displaystyle\int_1^\infty\frac{dx}{x^3} = \frac{1}{2}.$

(*c*) $\displaystyle\int_0^\infty\frac{dx}{(1+x)^{3/2}} = 2.$

(*b*) $\displaystyle\int_1^\infty\frac{dx}{\sqrt[3]{x}}$ is non-existent.

(*d*) $\displaystyle\int_0^\infty\frac{dx}{(1+x)^{2/3}}$ is non-existent.

(*e*) $\displaystyle\int_0^\infty\frac{dx}{(1+x)^n}$ is $\begin{cases}\text{determinate if } n > 1,\\ \text{non-existent if } n \leq 1.\end{cases}$

(*f*) $\displaystyle\int_0^\infty\frac{dx}{1+x^2} = \frac{\pi}{2}.$

(*i*) $\displaystyle\int_1^\infty\frac{\sqrt{1+x}}{x}\,dx$ is non-existent.

(*g*) $\displaystyle\int_a^\infty\frac{dx}{a^2+x^2} = \frac{\pi}{4a}.$

(*j*) $\displaystyle\int_0^\infty e^{-2x}\,dx = \tfrac{1}{2}.$

(*h*) $\displaystyle\int_1^\infty\frac{dx}{x^2\sqrt{x^2-1}} = 1.$

(*k*) $\displaystyle\int_0^\infty e^{2x}\,dx$ is non-existent.

4. Determine the area between each of the following curves, the x-axis, and the ordinates at the values of x indicated:

(a) $y^3(x-1)^2 = 1$; $x = 0$ to 9. *Ans.* 9.

(b) $xy^2(1+x)^2 = 4$; $x = 0$ to 4. *Ans.* $4\tan^{-1} 2$.

(c) $y^2x^4(1+x) = 1$; $x = 0$ to 3. *Ans.* ∞.

(d) $x^2y^2(x^2-1) = 9$; $x = 1$ to 2. *Ans.* 2π.

(e) $y^3(x-1)^2 = 8x^3$; $x = 0$ to 3. *Ans.* $9\sqrt[3]{2} + 9/2$.

(f) $x^2y^2(x^2+9) = 1$; $x = 4$ to ∞. *Ans.* $\frac{1}{3}\log 2$.

(g) $y^2(1+x)^4 = x$; $x = 0$ to ∞. *Ans.* π.

(h) $y^3(x+1)^2 = 1$; $x = 0$ to ∞. *Ans.* ∞.

5. If each of two curves $y = f(x)$ and $y = \phi(x)$ is asymptotic to the y-axis, and if $f(x) \geqq \phi(x) \geqq 0$, show that $\int_0^1 f(x)dx$ cannot exist unless $\int_0^1 \phi(x)\,dx$ exists. Hence show that $\int_0^1 x^{-2}\,dx$ does not exist by comparing it with $\int_0^1 x^{-1}\,dx$.

6. If each of two curves $y = f(x)$ and $y = \phi(x)$ is asymptotic to the x-axis, and if $f(x) \geqq \phi(x) \geqq 0$, show that $\int_k^\infty f(x)dx$ cannot exist unless $\int_k^\infty \phi(x)\,dx$ exists.

7.* Verify each of the following results (see *Tables*, IV, F and V, F):

(a) $\displaystyle\int_0^\infty e^{-x}\,dx = 1.$ (c) $\displaystyle\int_0^\infty x^2 e^{-x}\,dx = 2.$ (e) $\displaystyle\int_0^\infty x^4 e^{-x}\,dx = 4\,!$

(b) $\displaystyle\int_0^\infty xe^{-x}\,dx = 1.$ (d) $\displaystyle\int_0^\infty x^3 e^{-x}\,dx = 3\,!$ (f)* $\displaystyle\int_0^\infty x^n e^{-x}\,dx = n\,!$

8.* Show by means of Exs. 6 and 7 that $\int_0^\infty x^{1.5}e^{-x}\,dx$ exists. Find its value from the tables (IV, F, 109 and V, F). Find the value of $\int_0^\infty x^{2.5}e^{-x}\,dx$; the value of $\int_0^\infty x^{1.3}e^{-x}\,dx$; the value of $\int_0^\infty x^{3.4}e^{-x}\,dx$.

9. If $x^n \geqq f(x) \geqq 0$ for large values of x, where n is a positive integer, show that $\int_k^\infty f(x)e^{-x}\,dx$ exists. Hence show that $e^x > x^n$ for large values of x by showing that $\int_k^\infty e^x \cdot e^{-x}\,dx$ does not exist.

113. Repeated Integration. Repeated integrations may be performed with no new principles. Thus

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + c; \quad \text{and} \quad \int\left(-\frac{1}{x} + c\right)dx = -\log x + cx + c'.$$

The final answer might be called *the second integral* of $1/x^2$. Such processes are frequently used in solving differential equations (see § 92, and Chapter X).

Thus, in the case of a falling body, the tangential acceleration is constant:

$$j_T = \frac{dv}{dt} = -g,$$

where g is the constant; hence

$$v = \int j_T dt + \text{const.} = -gt + c;$$

but since $v = ds/dt$,

$$s = \int v\,dt + \text{const.} = \int (-gt + c)\,dt + \text{const.} = -\frac{gt^2}{2} + ct + c'.$$

If the body falls from a height of 100 ft., with an initial speed zero, $s = 100$ and $v = 0$ when $t = 0$; hence $c = 0$ and $c' = 100$, whence we find $s = -gt^2/2 + 100$.

The equations $s = \int v\,dt + \text{const.}$, $v = \int j_T dt + \text{const.}$, just obtained, apply in any motion problem, where j_T is the *tangential acceleration*, v is the *speed*, and s is the *distance* passed over. Substituting for v, we might write

$$s = \int \left[\int j_T dt + c\right] dt + c'.$$

114. Successive Integration in Two Letters. Another distinctly different case of repeated integration which can be performed without further rules is that in which the second integration is performed with respect to a different letter.

Thus, the volume of any solid is (§ 70, p. 121),

$$V = \int_{h=a}^{h=b} A_S\,dh, \tag{1}$$

where A_S is the area of a section perpendicular to the direc-

tion in which h is measured, and where $h = a$ and $h = b$ denote planes which bound the solid.

In many cases it is convenient first to find A_S by a first integration, by the methods of § 60, p. 103, and then integrate A_S to find V by (1), this second integration being with respect to the height h.

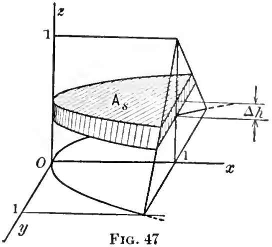


FIG. 47

Example 1. Find the volume of the parabolic wedge

$$y^2 = x(1 - z)^2$$

between the planes $z = 0$ and $z = 1$ and between the planes $x = 0$ and $x = 1$.

The area A_S of a section by any plane $z = h$ parallel to the xy-plane is twice the area between the curve $y = (1 - h)\sqrt{x}$ and the x-axis:

$$A_S\Big]_{x=0}^{x=1} = 2\int_{x=0}^{x=1} y\,dx = 2\int_{x=0}^{x=1}(1-h)\sqrt{x}\,dx = \frac{4}{3}(1-h)x^{3/2}\Big]_{x=0}^{x=1} = \frac{4}{3}(1-h),$$

hence this volume, by (1), is

$$V\Big]_{h=0}^{h=1} = \int_{h=0}^{h=1} A_S\,dh = \frac{4}{3}\int_{h=0}^{h=1}(1-h)\,dh = \frac{4}{3}\left(h - \frac{h^2}{2}\right)\Big]_{h=0}^{h=1} = \frac{2}{3}.$$

Notice that h, during the first integration, was essentially constant. Notice also that the volume of the wedge is one third the volume of the circumscribed rectangular parallelopiped; and that since A_S is a linear function of h, the *prismoid rule* (§ 71, p. 125) gives the volume precisely.

Combining the formulas used in this example, the volume V may be written

$$V\Big]_{h=0}^{h=1} = \int_{h=0}^{h=1}\left[2\int_{x=0}^{x=1}(1-h)\sqrt{x}\,dx\right]dh = \frac{2}{3}.$$

Such successive integrals in two letters are very common in all applications.

EXERCISES XLV. — SUCCESSIVE INTEGRATION

1. Determine a function $y = f(x)$ whose second derivative d^2y/dx^2 is $6x$. *Ans.* $y = x^3 + C_1x + C_2$.

2. Determine the speed v and the distance s passed over by a particle whose tangential acceleration d^2s/dt^2 is $6t$. Find the values of the arbitrary constants if $v=0$ and $s=0$ when $t=0$; if $v=100$ and $s=0$ when $t=0$.

3. Find the general expressions for functions whose derivatives have the following values:

(a) $d^2y/dx^2 = 6x^2$. (d) $d^2r/d\theta^2 = 1/\sqrt{1-\theta}$. (g) $d^2y/dx^2 = e^x$.
(b) $d^2s/dt^2 = 1 + 2t$. (e) $d^3r/d\theta^3 = \theta^2 - 2\theta$. (h) $d^2s/dt^2 = \sec^2 t$.
(c) $d^2s/dt^2 = \sqrt{1-t}$. (f) $d^3v/du^3 = 1 + u^2$. (i) $d^3v/du^3 = 1/u^2$.

4. Determine the speed v and distance s passed over in time t, when the tangential acceleration j_T and initial conditions are as below:

(a) $j_T = \sin t$; $v = 0$ and $s = 0$ when $t = 0$.
(b) $j_T = t + \cos t$; $v = 0$ and $s = 0$ when $t = 0$.
(c) $j_T = \sqrt{1+t}$; $v = 3$ and $s = 0$ when $t = 0$.
(d) $j_T = t/\sqrt{1+t^2}$; $v = 1$ and $s = 0$ when $t = 0$.

5. Evaluate each of the following integrals, taking the inner integral sign with the inner differential:

(a) $\int_{x=0}^{x=1}\int_{y=0}^{y=1} xy\,dy\,dx$. (f) $\int_{t=1}^{t=3}\int_{s=-2}^{s=0}\int_{r=0}^{r=2}\frac{r^2s^2}{t^3}\,dr\,ds\,dt$.

(b) $\int_{x=0}^{x=2}\int_{y=1}^{y=2} 6x^2(1-y)\,dy\,dx$. (g) $\int_{x=-1}^{x=+1}\int_{y=-x^2}^{y=+x^2}(x+y)\,dy\,dx$.

(c) $\int_{x=0}^{x=2}\int_{y=2}^{y=3}(x^2+1)(4-y^2)\,dy\,dx$. (h) $\int_{x=0}^{x=3}\int_{y=1}^{y=2x+3}(x+y)^2\,dy\,dx$.

(d) $\int_{v=2}^{v=4}\int_{u=1}^{u=2}\sqrt{u+v}\,du\,dv$. (i) $\int_0^1\int_{-x}^{+x}\int_0^{x+y}(x+y+z)\,dz\,dy\,dx$.

(e) $\int_{x=1}^{x=e}\int_{y=0}^{y=\pi}\frac{\sin y}{x}\,dy\,dx$. (j) $\int_0^r\int_0^\pi\int_0^\pi r^2\sin\theta\,d\theta\,d\phi\,dr$.

6. Find the volume of the part of the elliptic paraboloid $4x^2+9y^2=36z$ between the planes $z = 0$ and $z = 1$; between the planes $z = a$ and $z = b$.

7. Find the volume of the part of the cone $4x^2 + 9y^2 = 36z^2$ between the planes $z = 0$ and $z = 2$; between $z = a$ and $z = b$.

8. Find the volume of the part of the cylinder $x^2 + y^2 = 25$ between the planes $z = 0$ and $z = x$; between the planes $z = x/2$ and $z = 2x$.

9. A parabola, in a plane perpendicular to the x-axis and with its axis parallel to the z-axis, moves with its vertex along the x-axis. Its latus rectum is always equal to the x-coördinate of the vertex. Find the volume inclosed by the surface so generated, from $z = 0$ to $z = 1$ and from $x = 0$ to $x = 1$.

10. Find the volume of the part of the cylinder $x^2 + y^2 = 9$ lying within the sphere $x^2 + y^2 + z^2 = 16$.

11. For a beam of constant strength the deflection y is given by the fact that the flexion is constant: $b = d^2y/dx^2 =$ const. if the beam is of uniform thickness. Find y in terms of x and determine the arbitrary constants if $y = 0$ when $x = \pm l/2$. [This will occur if the beam is of length l, and is supported freely at both ends.]

12. Determine the arbitrary constants in the case of the beam of Ex. 11, if $y = 0$ and $dy/dx = 0$ when $x = 0$. [This will occur if the beam is rigidly embedded at one end.]

13. For a beam of uniform cross section loaded at one end and rigidly embedded at the other, $b = d^2y/dx^2 = k(l - x)$ where l is the length of the beam, x is the distance from one end, and k is a known constant which is determined by the load and the cross section of the beam. Find y in terms of x, and determine the arbitrary constants.

14. Find y in terms of x in each of the following cases:

(*a*) $d^2y/dx^2 = k(l^2 - 2lx + x^2)$; $y = 0$, $dy/dx = 0$ when $x = 0$.

[Beam rigidly embedded at one end, loaded uniformly.]

(*b*) $d^2y/dx^2 = a + bx$; $y = 0$, $dy/dx = 0$ when $x = 0$.

[Beam of uniform strength of thickness proportional to $(a + bx)^{-1}$, embedded at one end.]

(*c*) $d^2y/dx^2 = k(l^2/8 - x^2/2)$; $y = 0$ when $x = \pm l/2$.

[Beam supported at both ends, loaded uniformly.]

(*d*) $d^2y/dx^2 = k/x^2$; $y = 0$, $dy/dx = 0$ at $x = l$.

[Beam of uniform strength of thickness proportional to x^2, embedded at $x = l$.]

15. Find the angular speed ω and the total angle θ through which a wheel turns in time t, if the angular acceleration is $\alpha = d^2\theta/dt^2 = 2t$, and if $\theta = \omega = 0$ when $t = 0$.

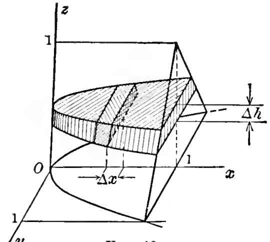


FIG. 48

115. Double Integrals. It is often convenient to restate such problems as that solved in § 114 in somewhat different form.

In obtaining the area A_s we originally (§ 66, p. 115) cut the area into strips of width Δx; their length is $2y$ each, since they reach from one side of the parabola to the other. We then showed that

$$(1)\qquad A_S = 2\int_{x=0}^{x=1} y\,dx = 2\lim_{\Delta x \doteq 0}\ [\text{Sum of terms like } y\,\Delta x.]$$

We may as well proceed to set up both integrations at once, as follows: let us consider the small column whose face is $2y\,\Delta x$ and whose thickness is Δh; its volume is

$$(2)\qquad 2\,y\,\Delta x\,\Delta h;$$

the volume of the whole layer whose base is A_s is

$$(3)\qquad A_s\cdot\Delta h = 2\,\Delta h\cdot\int_{x=0}^{x=1} y\,dx = 2\,\Delta h\cdot\lim_{\Delta x\doteq 0}\sum_{x=0}^{x=1}(y\,\Delta x),$$

where Σ stands for "the sum of terms like"; hence

$$(4)\qquad A_s\cdot\Delta h = 2\lim_{\Delta x\doteq 0}\Delta h\cdot\sum_{x=0}^{x=1}(y\,\Delta\,x) = 2\lim_{\Delta x\doteq 0}\sum_{x=0}^{x=1}(y\,\Delta x\,\Delta h).$$

The entire volume is, however (§ 70, p. 121):

$$(5)\qquad V\Big]_{h=0}^{h=1} = \int_{h=0}^{h=1} A_s\,dh = \lim_{\Delta h\doteq 0}\sum_{h=0}^{h=1} A_s\,\Delta h$$

$$= 2\lim_{\Delta h\doteq 0}\sum_{h=0}^{h=1}\lim_{\Delta x\doteq 0}\sum_{x=0}^{x=1}(y\,\Delta x\,\Delta h).$$

The expression

$$\lim_{\Delta h \doteq 0} \sum_{h=0}^{h=1} \lim_{\Delta x \doteq 0} \sum_{x=0}^{x=1} (y \, \Delta x \, \Delta h)$$

which occurs in (5) is equal to the double limit which follows; it is called a **double integral**, and is denoted by $\iint y \, dx \, dh$:

$$(6) \qquad \lim_{\substack{\Delta x \doteq 0 \\ \Delta h \doteq 0}} \sum_{h=0}^{h=1} \sum_{x=0}^{x=1} y \, \Delta x \, \Delta h = \int_{h=0}^{h=1} \int_{x=0}^{x=1} y \, dx \, dh.$$

In the particular example in hand, $y = (1 - h)\sqrt{x}$, and the limits are $(h = 0, h = 1)$ and $(x = 0, x = 1)$; but the argument was not affected by our knowledge of these values. It follows that the successive integrals mentioned in § 114 are always equal to the double limit in (6), where y is any function $F(x, h)$ of x and h we please:

$$(7) \quad \int_{h=a}^{h=b} \left[\int_{x=c}^{x=d} F(x, h) \, dx \right] dh = \lim_{\substack{\Delta x \doteq 0 \\ \Delta h \doteq 0}} \sum_{h=a}^{h=b} \sum_{x=c}^{x=d} y \, \Delta x \, \Delta h$$

$$= \int_{h=a}^{h=b} \int_{x=c}^{x=d} F(x, h) \, dx \, dh.$$

This is the **fundamental summation formula for double integrals**. In writing it, any letters, not necessarily x and h, may be used. Moreover c and d may depend on h, as we shall see in numerous examples. It is used exactly as we have used the original summation formula: quantities we desire to measure often appear most naturally in the forms of approximate double sums like (6). The accurate evaluation is done by successive integration by means of (7).

116. Illustrative Examples. In this paragraph, several applications of double integration are worked out. These should not be memorized, but rather the formulas should be built up by the student each time they are used.

[A] Volumes by Double Integration.* A problem which is essentially the same as that of § 114 is to find the volume under any surface whose equation is given in the form $z = F(x, y)$, where $F(x, y)$ is any function of x and y. Consider for example the volume V bounded by the surface, xz-plane, the planes $x=a$ and $x=b$ and the right cylinder whose base is a given curve $y=f(x)$ of the xy-plane.

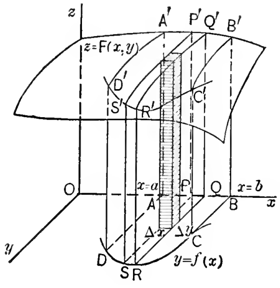


Fig. 49

If the volume is divided into layers by planes parallel to the yz-plane, equally placed at intervals Δx; and if these layers are themselves divided into small columns of width Δy, the volume of any one column is approximately $z\,\Delta y\,\Delta x$, and the total volume is

$$V = \lim_{\substack{\Delta x \doteq 0 \\ \Delta y \doteq 0}} \sum_{x=a}^{x=b} \sum_{y=0}^{y=f(x)} z\,\Delta y\,\Delta x = \int_{x=a}^{x=b} \int_{y=0}^{y=f(x)} F(x, y)\,dy\,dx.$$

Thus the volume under the surface $z = x^2 + y^2$ between the xz-plane, the planes $x = 0$ and $x = 1$, and the cylinder whose base is $y = \sqrt{x}$ is

$$V = \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{x}} (x^2 + y^2)\,dy\,dx = \int_{x=0}^{x=1} \left[x^2 y + \frac{y^3}{3}\right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_{x=0}^{x=1} \left(x^{5/2} + \frac{1}{3}x^{3/2}\right) dx = \frac{44}{105}.$$

[B] Area in Polar Coördinates. The area A bounded by a curve whose equation in polar coördinates is $\rho = f(\theta)$, and two radii vectors $\theta = \alpha$, $\theta = \beta$ is approximated by dividing it into triangular strips by radii vectors spaced at equal angles $\Delta\theta$. If we then draw circles with centers at O, equally spaced at intervals $\Delta\rho$, the whole area A is divided into small curvilinear "squares" like the one shaded in Fig. 50.

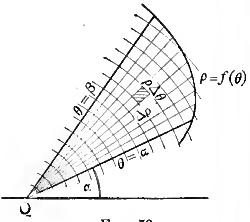


Fig. 50

* Formulas from Solid Analytic Geometry are to be found in Chapter IX.

The straight line side of one of these is $\Delta\rho$, while the circular side has a length $\rho\Delta\theta$, where ρ is the value of ρ along that side. Hence the area of the shaded "square" is, approximately, $\rho\,\Delta\rho\,\Delta\theta$ and the area to be found is, precisely,

$$A = \lim_{\substack{\Delta\rho \doteq 0 \\ \Delta\theta \doteq 0}} \sum_{\theta=\alpha}^{\theta=\beta} \sum_{\rho=0}^{\rho=f(\theta)} \rho\,\Delta\rho\,\Delta\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=0}^{\rho=f(\theta)} \rho\,d\rho\,d\theta.$$

The first integration $\int \rho\,d\rho$ can always be performed, since $\int \rho\,d\rho = \rho^2/2$; but it is best not to burden the memory * with this, since it is evident each time such an area is to be found. Thus the area bounded by the curve $\rho = \sec\theta$ (draw it) and the lines $\theta = 0$, $\theta = \pi/4$ is

$$\int_{\theta=0}^{\theta=\pi/4} \int_{\rho=0}^{\rho=\sec\theta} \rho\,d\rho\,d\theta = \int_{\theta=0}^{\theta=\pi/4} \left[\frac{\rho^2}{2}\right]_{\rho=0}^{\rho=\sec\theta} d\theta = \int_{\theta=0}^{\theta=\pi/4} \frac{\sec^2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\tan\theta\,\Big]_{\theta=0}^{\theta=\pi/4} = \frac{1}{2}.$$

[*C*] Moment of Inertia of a Thin Plate. The moment of inertia *I* about a point *O* of a small object whose mass is m is defined in Physics to be the product of the mass times the square of the distance from O to the object: $I = mr^2$.

Given now a thin plate of metal of uniform density and thickness, whose boundary C is a given curve, let us divide the plate into small squares by lines equally spaced parallel to two rectangular axes through O. Let P be a point in any one of these squares and let $OP = r = \sqrt{x^2 + y^2}$. Then the mass of the square is $k \cdot \Delta y\,\Delta x$ where k denotes the constant surface density (*i.e.* the mass per square unit); and the moment of inertia of this square about O is, approximately, $k \cdot r^2\,\Delta y\,\Delta x$. Hence the moment of inertia I of the entire plate about O is:

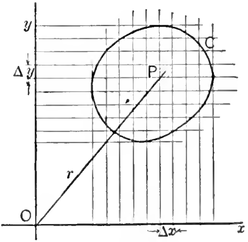


FIG. 51

$$I = \lim_{\substack{\Delta y \doteq 0 \\ \Delta x \doteq 0}} \sum\sum kr^2\,\Delta x\,\Delta y = \int\int (x^2 + y^2)\,dy\,dx,$$

* If any part of this work is memorized, it should be at most the *figure* drawn above.

where proper limits of integration are to be inserted to cover the area enclosed by C. If C is an oval, as shown in the figure, the limits of y are the values of y along the lower half oval and the upper half; these must be given in the problem as functions of x. The limits for x are the extreme values of x on the two ends of the oval.

Thus the moment of inertia of a plate bounded by the two curves $y = (1 - x^2)$ and $y = (x^2 - 1)$, about the origin (draw the figure) is:

$$I = k \int_{x=-1}^{x=+1} \int_{y=x^2-1}^{y=1-x^2} (x^2 + y^2) dy\, dx = k \int_{x=-1}^{x=+1} \left[x^2 y + \frac{y^3}{3} \right]_{y=x^2-1}^{y=1-x^2} dx$$

$$= \frac{2k}{3} \int_{x=-1}^{x=+1} (1 - x^6)\, dx = \frac{2k}{3} \left[x - \frac{x^7}{7} \right]_{x=-1}^{x=+1} = \frac{8}{7} k,$$

where k is the surface density.

[D] **Moment of Inertia in Polar Coördinates.** Using the figure drawn for [B], it is easy to see that the moment of inertia of a thin plate of the shape of the area in [B] is:

$$I = \lim_{\substack{\Delta\rho \doteq 0 \\ \Delta\theta \doteq 0}} k \cdot \sum_{\theta=\alpha}^{\theta=\beta} \sum_{\rho=0}^{\rho=f(\theta)} \rho^3\, \Delta\rho\, \Delta\theta = k \cdot \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=0}^{\rho=f(\theta)} \rho^3\, d\rho\, d\theta,$$

where k is the surface density (*i.e.* mass per unit area) as in [C].

Thus for a circle whose center is O, $\rho = f(\theta) = a$, the radius. Hence, the moment of inertia of a circular disk about its center is:

$$I = k \cdot \int_{\theta=0}^{\theta=2\pi} \left[\frac{\rho^4}{4} \right]_{\rho=0}^{\rho=a} d\theta = k \cdot \int_{\theta=0}^{\theta=2\pi} \frac{a^4}{4}\, d\theta = \pi k \frac{a^4}{2} = \frac{Ma^2}{2},$$

where k is the surface density, and $M = k\pi a^2$ is the mass of the disk.

EXERCISES XLVI. — DOUBLE INTEGRALS

1. Find the volume under the surface $z = x^2 + y^2$ between the xz-plane, the planes $x = 0$ and $x = 1$, and the cylinder whose base is the curve $y = x^2$.

2. Find the volume between the xy-plane and each of the following surfaces cut off by the planes and surfaces mentioned in each case:

(*a*) $z = x + y$ cut off by $y = 0$, $x = 0$, $x = 1$, $y = \sqrt{x}$.

(*b*) $z = x^2 + y$ cut off by $y = 0$, $x = 1$, $x = 3$, $y = x^2$.

(*c*) $z = xy$ cut off by $y = 0$, $x = 2$, $x = 4$, $y = x^2 + 1$.

(*d*) $z = xy + y^2$ cut off by $y = 0$, $x = 1$, $x = 5$, $y = x^3$.
(*e*) $z = y + \sqrt{x}$ cut off by $y = 0$, $x = 0$, $x = 1$, $y = x^5$.
(*f*) $z = x^2 + y^3$ cut off by $x = 0$, $y = 1$, $y = 4$, $y^2 = x$.
(*g*) $z = \sqrt{x + y}$ cut off by $x = 0$, $y = 2$, $y = 5$, $y = x$.
(*h*) $z = x^2 + 4y^2$ cut off by $y = 0$ and $y = 1 - x^2$.
(*i*) $z = xy$ cut off by $y = x^2$ and $y = 1$.
(*j*) $z = x^2 - y^2$ cut off by $y = x^2$ and $y = x$.
(*k*) $z = xy - y^2$ cut off by $y = x^2$ and $y = 2 - x^2$.

3. Find the volume of the portion of the paraboloid $z = 1 - x^2 - 4y^2$ which lies in the first octant.

4. If two plane cuts are made to the same point in the center of a circular cylindrical log, one perpendicular to the axis and the other making an angle of 45° with it, what is the volume of the wedge cut out?

5. Show that the volume common to two equal cylinders of radius a which intersect centrally at right angles is $16a^3/3$.

6. Show that the volume of the ellipsoid $x^2/16 + y^2/9 + z^2/4 = 1$ is 32π.

7. What part of the ellipsoid in Ex. 6 lies within a cube whose center is at the origin and whose edges are 6 units long and parallel to the coördinate axes?

8. Where should a plane perpendicular to the x-axis be drawn so as to divide the volume of the ellipsoid in Ex. 6 in the ratio 2 : 1?

9. Calculate by double integration the areas bounded by the following curves:

(*a*) $y = x^2$ and $y = \sqrt{x}$.
(*b*) $y = x^2$ and $y = x^3$.
(*c*) $y = x^2$ and $-x^2 + y^2 = 2$.
(*d*) $x^2 + y^2 = 12$ and $y = x^2$.
(*e*) $x = 0$, $y = \sin x$, and $y = \cos x$.
(*f*) $y = 0$, $y^2 = x$, and $x^2 - y^2 = 2$.
(*g*) $y = 2x$, $y = 0$, and $y = 1 - x$.
(*h*) $y^2 = x$, and $y = 1 - x$.

10. Calculate the moment of inertia of a thin plate bounded by the curves $y = x^2$, $y = 2 - x^2$, about the origin.

11. Calculate the moment of inertia of a thin plate about the origin, in each of the cases in which the shape of the plate is the area bounded by the curves in one of the parts of Ex. 9.

12. Find the moment of inertia of each of the following shapes of thin plate:

(*a*) A square about a diagonal. About a corner.

(*b*) A right triangle about a side. About the vertex of the right angle.

(*c*) A circle about its center.

(*d*) An ellipse about either axis. About the center.

(*e*) A circle about a diameter.

(*f*) A trapezoid about a line parallel to its parallel sides.

13. Find the moment of inertia of a thin spoke of a wheel about the center of the wheel.

14. Determine the entire area, or the specified portion of the area, bounded by each of the following curves, whose equations are given in polar coördinates:

(*a*) $\rho = 2\cos\theta$. *Ans.* π.

(*b*) One loop of $\rho = \sin 2\theta$. *Ans.* $\pi/8$.

(*c*) One loop of $\rho = \sin 3\theta$. *Ans.* $\pi/12$.

(*d*) The cardioid $\rho = 1 - \cos\theta$. *Ans.* $3\pi/2$.

(*e*) The lemniscate $\rho^2 = \cos 2\theta$. *Ans.* 1.

(*f*) The spiral $\rho = \theta$ from $\theta = 0$ to π. *Ans.* $\pi^3/6$.

(*g*) The spiral $\rho\theta = 1$ from $\theta = \pi/4$ to $\pi/2$. *Ans.* $1/\pi$.

(*h*) $\rho = 1 + 2\cos\theta$ from $\theta = 0$ to π. *Ans.* $3\pi/2$.

(*i*) $\rho = \tan\theta$ from $\theta = 0$ to $45°$. *Ans.* $1/2 - \pi/8$.

(*j*) The area between the nth and $(n+1)$th turns of each of the spirals in Exs. 14 (*f*), 14 (*g*).

15. Calculate the moment of inertia of a thin plate about the origin, for each of the shapes defined by the areas mentioned in Exs. 14 (*a*)–(*i*).

16. Calculate the following moments of inertia:

(*a*) A thin circular plate, about its center.

(*b*) A thin circular plate, about a point on the circumference.

(*c*) A thin plate bounded by two concentric circles, about the center.

(*d*) An equilateral triangle, about its center.

(*e*) An equilateral triangle, about one vertex.

17. The square of the **radius of gyration** ρ_g of a body, about any point, is its moment of inertia about that point divided by its mass: $\rho_g{}^2 = I \div M$. Find the radius of gyration for the example solved in [*C*], § 116; in [*D*], § 116.

18. Find the radius of gyration for each of the thin plates described in Exs. 9, 10, 12, 14, 16.

19. If $f(x, y)$ is any function of x and y, its **average** over a region is

$$\textbf{Average of } f(x, y) = \int\int f(x, y)\,dx\,dy \div \int\int dx\,dy.$$

Show that the square of the radius of gyration about the origin of a thin plate is the average value of $r^2 = x^2 + y^2$ over the surface of the plate.

20. Find the average value of x over the area described in [C], § 116, p. 214. Find the average value of y over the same area.

[NOTE. The point whose coördinates are the averages of values of x and y over an area is called the **center of gravity** or **centroid** of that area.]

21. Find the centroids of each of the areas mentioned in Exs. 9 and 14.

22. Find, for the area mentioned in [C], § 116, p. 214, the average value of each of the following functions:

(a) xy. (b) $x^2 + 4y^2$. (c) $x + y$. (d) $x^2 - y^2$.

117. Triple and Multiple Integrals. There is no difficulty in extending the ideas of §§ 113–116 to threefold integrations or to integrations of any order. Following the same reasoning, it is possible to show that, if $w = F(x, y, z)$

$$\lim_{\substack{\Delta x \doteq 0\\ \Delta y \doteq 0\\ \Delta z \doteq 0}} \sum_{z=e}^{z=f} \sum_{y=c}^{y=d} \sum_{x=a}^{x=b} w\,\Delta x\,\Delta y\,\Delta z$$

$$= \int_{z=e}^{z=f} \int_{y=c}^{y=d} \int_{x=a}^{x=b} F(x, y, z)\,dx\,dy\,dz,$$

where the three integrations are to be carried out in succession, where the limits for x may depend on y and z, and where the limits for y may depend on z: but the limits for z are, of course, constants.

Thus it is readily seen that the volume mentioned in (A), § 116, may be computed by dividing up the entire volume by three sets of equally spaced planes parallel to the three coördinate planes. Then the total volume is, approximately, the sum of a large number of cubes, the volume of each of which

is $\Delta x\,\Delta y\,\Delta z$; and its exact value is

$$V = \lim_{\substack{\Delta x \doteq 0\\ \Delta y \doteq 0\\ \Delta z \doteq 0}} \sum_{x=a}^{x=b} \sum_{y=0}^{y=f(x)} \sum_{z=0}^{z=F(x,\,y)} \Delta z\,\Delta y\,\Delta x$$

$$= \int_{x=a}^{x=b} \int_{y=0}^{y=f(x)} \int_{z=0}^{z=F(x,\,y)} dz\,dy\,dx,$$

which reduces to the result of [A], § 116, if we note that

$$\int_{z=0}^{z=F(x,\,y)} dz = z\Big]_{z=0}^{z=F(x,\,y)} = F(x,\,y).$$

Likewise the moment of inertia I (see § 116, [C]) of the same volume with respect to the origin is approximately the sum of terms of the sort $k(x^2 + y^2 + z^2)\,\Delta x\,\Delta y\,\Delta z$ where k is the density (mass per unit volume); whence the exact value of I is *

$$I = k \cdot \int_{x=a}^{x=b} \int_{y=0}^{y=f(x)} \int_{z=0}^{z=F(x,\,y)} (x^2 + y^2 + z^2)\,dz\,dy\,dx.$$

EXERCISES XLVII. — MULTIPLE INTEGRALS

1. Determine the volume bounded by the surface $z = (x + y)^2$, the coördinate planes, and the plane $x + y + z = 1$.

2. Write each of the volumes mentioned in Ex. 2, and in Exs. 3–6, List XLVI, as a triple integral; show that one integration reduces the triple integral to the double integral used before, in each instance.

3. Find the volume of the sphere by triple integration.

4. Write down the moment of inertia about the origin of each of the solids bounded by the surfaces mentioned in Ex. 2, and in Exs. 3–6, List XLVI. Actually carry out each of these integrations.

5. Write down the moment of inertia of a right cylinder of height l whose base is any one of the areas mentioned in Ex. 9, List XLVI, about an axis through the origin parallel to the elements of the cylinder. Show that one integration reduces the integral essentially to the double integral used in List XLVI, in each instance.

* It is well to urge that such formulas should *not* be remembered, but obtained in each exercise by the simple reasoning used above.

6. The square of *radius of gyration* of a solid about a point (or about a line) is the moment of inertia divided by the total mass. Find its value for each of the solids mentioned in Ex. 4, above; for each of the figures mentioned in Ex. 12, List XLVI.

7. Find the *average value*:

$$\text{Average of } f(x, y, z) = \int\int\int f(x, y, z)\, dx\, dy\, dz \div \int\int\int dx\, dy\, dz,$$

of each of the following functions, over the region mentioned in Ex. 1:

(*a*) $f(x, y, z) = x$. (*d*) $f(x, y, z) = xyz$. (*g*) $f(x, y, z) = x^2 + z^2$.

(*b*) $f(x, y, z) = y$. (*e*) $f(x, y, z) = xy$. (*h*) $f(x, y, z) = x^2 + y^2$.

(*c*) $f(x, y, z) = z$. (*f*) $f(x, y, z) = x^2 + y^2$. (*i*) $f(x, y, z) = x + y + z$.

[NOTE. The point whose coördinates are the three values given by (*a*), (*b*), (*c*), is called the **center of gravity**, or **centroid** of the volume.]

8. Find the centroid of the solid mentioned in Ex. 3, List XLVI.

9. Show that, in spherical coördinates (ρ, θ, ϕ), the volume of a solid is given by an integral of the form $\int\int\int \rho^2 \sin\theta\, d\theta\, d\phi\, d\rho$, where θ is the colatitude, and ϕ is the longitude, on a sphere of radius ρ.

10. Calculate the volume of a sphere by the integral in Ex. 9.

11. Calculate the volume cut from a circular cone by two concentric spheres with centers at the vertex of the cone.

12. Show that, in cylindrical coördinates (ρ, θ, z), the volume of a solid is given by an integral of the form $\int\int\int \rho\, d\theta\, d\rho\, dz$.

13. Calculate the volume of a sphere in cylindrical coördinates.

14. Determine the part of the cylinder $\rho = 2\sin\theta$ which lies between the planes $z = 0$ and $z = y$.

15. Determine the part of the cylinder $\rho = \sin 2\theta$ which lies between the planes $z = 0$ and $x + y + z = \sqrt{2}$.

118. Other Applications of Integration. Averages. Centers of Gravity. Among other applications of integration already mentioned in exercises, one very general idea is that of the

average value (A. V.) **of a quantity**:

$$\text{A. V. of } f(x)\Big]_{x=a}^{x=b} = \frac{\int_{x=a}^{x=b} f(x)\,dx}{b-a} = \frac{\int_{x=a}^{x=b} f(x)\,dx}{\int_{x=a}^{x=b} dx}$$

and analogous forms for functions defined in a given area or in a given volume. See § 71, p. 126; Ex. 19, p. 217; Ex. 7, p. 219; and *Tables*, IV, H, 138.

In particular, the **center of gravity**, or **centroid**, of an object is defined as the point such that each coördinate of that point is the *average* of the same coördinate throughout the body; thus, for a thin plate, the coördinates $(\bar{x}, \bar{y})$ of the center of gravity are

$$\bar{x} = \frac{\int\int kx\,dx\,dy}{\int\int k\,dx\,dy} = \frac{\int\int kx\,dx\,dy}{M},$$

$$\bar{y} = \frac{\int\int ky\,dx\,dy}{\int\int k\,dx\,dy} = \frac{\int\int ky\,dx\,dy}{M},$$

where k is the surface density and M is the total mass; and where the limits of integration to be inserted are the same as those inserted in finding the area of the plate. These formulas hold even when the density k is variable.

Similar formulas hold for centers of gravity of solids; for other averages, such as the center of water pressure on a dam; and for a variety of other scientific problems. A short list of these formulas is given in the *Tables*, IV, H. These may be used in solving exercises which follow.

EXERCISES XLVIII. — GENERAL PROBLEMS IN INTEGRATION

[These problems may be used for further drill and for reviews; it is advised that not all of them be done on first reading. Many of these may be reserved until after Chapter IX.]

1. Carry out the following integrations:

(a) $\int (1+t^2)^3\,dt.$ (b) $\int (t^{1/5}+t^2-c)\,dt.$ (c) $\int \frac{x^4dx}{a+bx^5}\cdot$ (d) $\int e^{a+bz}\,dz.$ (e) $\int \frac{u-a}{u-5u^2}\,du.$ (f) $\int \frac{d\theta}{\cos 2\theta}\cdot$ (g) $\int \frac{x\,dx}{(1-x)^5}\cdot$

(h) $\int \frac{dx}{x^3+3x^2}\cdot$ (i) $\int \frac{3x^4\,dx}{\sqrt{x^2+4x^6}}\cdot$ (j) $\int e^x \sin 2x\,dx.$ (k) $\int \sec x/2 \tan x/2\,dx.$ (l) $\int \cos^3 u \sin^3 u\,du.$ (m) $\int \frac{\log x+1}{x\log x+1}\,dx.$ (n) $\int \frac{x^2+4x-6}{x^3+6x^2}\,dx.$

(o) $\int e^{t^2}t^3\,dt.$ (p) $\int \frac{e^{2x}\,dx}{(e^x+2)^{1/5}}\cdot$ (q) $\int \frac{1+\operatorname{ctn}\theta}{\sin\theta}\,d\theta.$ (r) $\int \frac{u\,du}{\sqrt{1-u^2-u^4}}\cdot$ (s) $\int (\sec 2x+1)^2\,dx.$ (t) $\int 2^t\,dt.$ (u) $\int t\cos^{-1}t\,dt.$

2. Evaluate the following definite integrals; notice particularly whether the integral is improper, and if it is, explain your result:

(a) $\int_1^5 \frac{2\,dx}{1+4x^2}\cdot$ (b) $\int_2^{3.5} xe^{3x^2-1}\,dx.$ (c) $\int_0^{\pi} \tan t\,dt.$ (d) $\int_{-\pi}^{+\pi} \cos 2x \sin x\,dx.$ (e) $\int_1^e x^3 \log x^2\,dx.$ (f) $\int_{-\pi}^{+\pi} (\cos 2x)^2\,dx.$ (g) $\int_0^3 \frac{dx}{\sqrt{15x-5x^2}}\cdot$

(h) $\int_{-1}^{+1} \frac{x^3\,dx}{\sqrt{1-x^2}}\cdot$ (i) $\int_0^1 \frac{x+1}{x+\sqrt{x}}\,dx.$ (j) $\int_1^2 \frac{x\,dx}{\sqrt{1+x^4}}\cdot$ (k) $\int_1^5 \frac{\sqrt{n+1}}{\sqrt{n-2}}\,dn.$ (l) $\int_a^b \frac{dt}{at^2+2bt+c}\cdot$ (m) $\int_{-\pi}^{+\pi} \cos 2x \cos 3x\,dx.$ (n) $\int_{-\pi}^{+\pi} \cos mx \sin nx\,dx.$

(o) $\int_0^a \sqrt{1+4x^2}\,dx.$ (p) $\int_1^{\infty} \frac{dx}{(1+x^2)\tan^{-1}x}\cdot$ (q) $\int_2^3 \frac{dx}{(x+1)\sqrt{x^2-1}}\cdot$ (r) $\int_0^1 \frac{\sin x^{1/3}}{x^{2/3}}\,dx.$ (s) $\int_0^{\pi/2} x\cos x\,dx.$ (t) $\int_0^{\pi} \frac{d\psi}{a+b\cos\psi}\cdot$ (u) $\int_{-1}^{+1} \frac{p^5-p^3+p}{(p^2+16)^2}\,dp.$

3. By use of numerical tables, find the values of the following integrals:

(a) $\int_{1.2}^{3.45} \frac{dx}{1+x^2}$. (c) $\int_{.25}^{1.2} \frac{dt}{\cos^2 t}$. (e) $\int_1^{2.5} \frac{dx}{\sqrt{x^2-1}}$.

(b) $\int_{1.45}^{2.36} \frac{dx}{x+1}$. (d) $\int_1^2 e^{2x-3}\,dx$. (f) $\int_{-5}^{+5} \frac{dt}{\sqrt{t^2+9}}$.

(g) $\int_0^{45^\circ} \frac{d\theta}{\sqrt{1-.36\sin^2\theta}}$. (h) $\int_0^{30^\circ} \sqrt{4-\sin^2\theta}\,d\theta$.

(i) $\int_{30^\circ}^{60^\circ} \frac{dx}{\sqrt{5-2\sin^2 x}}$. (j) $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}\sqrt{4-x^2}}$.

(k) $\int_{.5}^1 \frac{e^x\,dx}{x}$. (m) $\int_e^{e^2} \frac{dx}{\log x}$. (o) $\int_0^\infty e^{-x}x^{1.7}\,dx$.

(l) $\int_0^1 e^{-x^2}\,dx$. (n) $\int_0^{\sqrt{2}/2} \sqrt{\frac{4-x^2}{1-x^2}}\,dx$. (p) $\int_0^{1/e} \frac{dx}{\log 2x}$.

4. Show that

$$\int_0^1 \frac{dx}{1+2x\cos\phi+x^2} = \frac{1}{2}\int_0^\infty \frac{dx}{1+2x\cos\phi+x^2} = \frac{\phi}{2\sin\phi},$$

and explain what occurs when $\phi = 0$, and when $\phi = \pi/2$.

5. Integrate the following general integrals; where $f'(x)$ denotes the derivative of $f(x)$.

(a) $\int f(x)f'(x)\,dx$. (d) $\int f'(2x)\,dx$.

(b) $\int e^{kf(x)}f'(x)\,dx$. (e) $\int \frac{f'(x)}{f(x)}\,dx$.

(c) $\int f'(\cos x)\sin x\,dx$. (f) $\int \frac{f'(x)\,dx}{1+[f(x)]^2}$.

6. Verify the result of integrating $\sin^5 x\,dx$ by comparing it with the integral of $\cos^5 u\,du$ by means of the substitution $u = \pi/2 - x$.

7. Evaluate each of the following integrals:

(a) $\int_0^{\pi/2}\int_{-\pi/2}^{\pi/2} \sin(u+v)\,du\,dv$. (b) $\int_{s=0}^{s=1}\int_{t=s}^{t=s^2} se^t\,dt\,ds$.

8. Calculate the area A, between the x-axis and the curve $y = x^3 - 9x^2 + 23x - 15$, from $x = 1$ to $x = 3$, by direct integration and also by Simpson's Rule. Find the centroid $(\bar{x}, \bar{y})$ of the same area.

9. Proceed as in Ex. 8 for each of the following curves, between the limits stated below:

(*a*) $y = 1 + x - x^2 + x^3$; $x = 0$ to $x = 2$.

(*b*) $y = a(1 - x^2/b^2)$; 1st quadrant.
Ans. $A = 2\,b/3$; $\bar{x} = 3\,b/8$, $\bar{y} = 2\,a/5$.

(*c*) $y = x/(1 + x^2)$; $x = 0$ to $x = 1$.
Ans. $A = (1/2)\log_e 2$; $\bar{x} = 0.6192$, $\bar{y} = 0.2059$.

(*d*) $y = (e^{ax} + e^{-ax})/2\,a = (1/a)\cosh ax$; $x = 0$ to $x = k$.

(*e*) The sine curve; one arch. *Ans.* $A = 2$; $\bar{x} = \pi/2$, $\bar{y} = \pi/8$.

(*f*) The cycloid; one arch. *Ans.* $A = 3\,\pi a^2$; $\bar{x} = a\pi$, $\bar{y} = 5\,a/6$.

(*g*) $x^{2/3} + y^{2/3} = a^{2/3}$ [*or* $x = a\cos^3 t$, $y = a\sin^3 t$]; first quadrant.
Ans. $3\,\pi a^2/32$; $\bar{x} = \bar{y} = 256\,a/(315\,\pi)$.

(*h*) $x = a\sin t + b\tan t$, $y = a\cos t$; $t = 0$ to $t = \pi/4$.
Ans. $A = a^2(\pi + 2)/8 + ab\log\tan(3\pi/8)$.

(*i*) $x = 2\,a\sin^2\phi$, $y = 2\,a\sin^2\phi\tan\phi$; between the curve and its asymptote. *Ans.* $A = 3\,\pi a^2$; $\bar{x} = 5\,a/3$; $\bar{y} = 0$.

10. Find the areas bounded by each of the following curves, or the part specified:

(*a*) $\rho = a\theta^2$; one turn. (*b*) $\rho = a\cos\theta + b$.

(*c*) $\rho = a\sin\theta\cos\theta/(\sin^3\theta + \cos^3\theta)$ [folium]; the loop. *Ans.* $a^2/6$.

(*d*) $\pm x + \sqrt{a^2 - y^2} = a\log[(a + \sqrt{a^2 - y^2})/y]$ [tractrix]; above $y = 0$.

(*e*) $y^2(a - x) = x^3$ [cissoid]; to its asymptote $x = a$.

11. Find the volume generated by revolving each of the following curves about the line specified:

(*a*) $y = 5\,x/(2 + 3\,x)$; about $y = 0$; $x = 0$ to $x = 1$. *Ans.* $1.5558\cdots$.

(*b*) $2\,x^2 + 5\,y^2 = 8$; about $y = 0$; total solid. *Ans.* $64\,\pi/15$.

(*c*) $y^m = ax^n$; about $y = 0$; $(0, 0)$ to (x, y). *Ans.* $mny^2x/(2\,n + m)$.

(*d*) $y = b\sin(x/a)$; about $y = 0$; $x = 0$ to $x = \pi$.

(*e*) $y = a\cosh(x/a)$; about $y = 0$; $x = 0$ to $x = a$.

(*f*) $(x - a)^2 + y^2 = r^2$; about $x = 0$; total solid.

(*g*) The cycloid; about base; one arch. *Ans.* $5\,\pi^2 a^3$.

(*h*) The cycloid; about tangent at maximum; one arch. *Ans.* πa^3.

(*i*) The tractrix; about asymptote; total. *Ans.* $2\,\pi a^3/3$.

(*j*) $x = a\sin t + b\tan t$, $y = a\cos t$; about $y = 0$; $t = 0$ to $t = t$.
Ans. $\pi[a^3(\sin t - (1/3)\sin^3 t) + a^2bt]$.

(*k*) $y^2(a - x) = x^3$; about asymptote; total solid. *Ans.* $2\,\pi^2 a^3$.

(*l*) $x = a\cos^3 t$, $y = a\sin^3 t$; about $y = 0$; total solid. *Ans.* $32\,\pi a^3/105$.

12. Obtain a formula for the volume of a spherical segment of height h.

13. Show that the volume of an ellipsoid of three unequal semiaxes, a, b, c, is $4\pi abc/3$.

14. Show that the volume bounded by the cylinder $x^2+y^2=ax$, the paraboloid $x^2+y^2=bz$, and the xy-plane is $(3/32)(\pi a^4/b)$.

15. Find the volume common to a sphere and a cone whose vertex lies on the surface and whose axis coincides with a diameter of the sphere.

16. Describe the solid whose volume is given by each of the following integrals; and calculate the volume:

(a) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{y+c} dz\,dy\,dx.$ (c) $\int_0^{2a} \int_0^{x} \int_y^{x} dz\,dy\,dx.$

(b) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{x+y} dz\,dy\,dx.$ (d) $\int_0^a \int_0^{\sqrt{a+x}} \int_0^{y} dz\,dy\,dx.$

17. Show that the x-coördinate of the **center of gravity**, or **centroid**, of any frustum of any solid is:

$$\bar{x} = \int_a^b x\left[\int\int dy\,dz\right]dx \div V = \int_a^b A_S\cdot x\,dx \div \int_a^b A_S\,dx,$$

if A_S is the area of a section perpendicular to the x-axis, V is the total volume, and $x=a$ and $x=b$ are the truncating planes. State similar formulas for $\bar{y}$ and $\bar{z}$.

[NOTE. The integral $\int\int\int x\,dx\,dy\,dz$ is often called the **moment** (or the **first** moment) of the solid about the yz plane.]

18. Find the centroid of each of the following frusta:

(a) Of the paraboloid $x^2+y^2=4az$ by the plane $z=c$. *Ans.* $\bar{z}=2c/3$.

(b) Of a hemisphere. *Ans.* $\bar{z}=3r/8$.

(c) Of the upper half of the ellipsoid of revolution $4x^2+4y^2+9z^2=36$.

(d) Of the upper half of the ellipsoid $x^2+4y^2+9z^2=36$.

(e) Of the solid of revolution formed by revolving half of one arch of a cycloid about its base. *Ans.* $\bar{x}=\pi a/2+64a/(45\pi)$.

19. Show that, if A_S is any quadratic function of x, in Ex. 17, the moment of the volume about the yz plane is

$$\bar{x}\cdot V = [aB + bT + 2(b-a)M](b-a)/6,$$

where B, T, M denote, respectively, the areas of cross sections by $x=a$, $x=b$, $x=(b-a)/2$. [Compare § 71, p. 126, and Ex. 3, p. 128.]

20. Find the lengths of the arcs of each of the following curves, between the points specified:

(*a*) $y = \log x$; $x = a$ to $x = b$. *Ans.* $\left[\sqrt{1+x^2} - \log\{(\sqrt{1+x^2}+1)/x\}\right]_a^b$.

(*b*) $e^y \cos x = 1$; $x = 0$ to $x = x$.

(*c*) $x = t^2$, $y = 2\,at$ (or $y^2 = 4\,a^2x$); $t = t_1$ to $t = t_2$.

Ans. $\left[t\sqrt{a^2+t^2} + a^2 \log(t + \sqrt{a^2+t^2})\right]_{t_1}^{t_2}$.

(*d*) One arch of a cycloid. *Ans.* $8\,a$.

(*e*) $\rho = a(1 + \cos\theta)$ [cardioid]; total length. *Ans.* $8\,a$.

21. Find the moment of inertia and the radius of gyration (Ex. 6, List XLVII) of each of the areas mentioned in Ex. 9, about the origin.

22. Calculate the moment of inertia I for a right circular cone about its axis. *Ans.* (3/10) mass · square of radius.

23. Calculate the moment of inertia and the radius of gyration for the rim of a flywheel about its axis, the inner and outer radii being R_1, R_2.

Ans. Mass $(R_1^2 + R_2^2)/2$, $\sqrt{(R_1^2 + R_2^2)/2}$.

24. The moment of inertia of an ellipsoid about any one of its axes is (1/5) (mass) (sum of the squares of the other two semi-axes).

25. Calculate the moment of inertia for a spherical segment about the axis of the segment.

26. Show that, for any body, $2\,I_0 = I_x + I_y + I_z$, where I_0, I_x, I_y, I_z denote respectively its moments of inertia about a point and three rectangular axes through that point.

27. Show that for any figure in the xy-plane, $I_z = I_x + I_y$, where I_x, I_y, I_z denote its moments of inertia about the three coördinate axes respectively.

28. Show that the total pressure on a rectangle of height h feet and width b feet immersed vertically in water so that its upper edge is a feet below the surface and parallel to it, is $62.4\,bh(a + h/2)$. Show that the depth of the center of pressure is at $(6\,a^2 + 6\,ah + 2\,h^2)/(6\,a + 3\,h)$.

29. Show that the total pressure on a circle of radius r, immersed vertically in water so that its center is at a depth $a+r$, is $62.4\,\pi r^2(a+r)$. Show that the depth of the center of pressure is $a + r + r^2/(4\,r + 4\,a)$.

30. Show that the total pressure on a semicircle, immersed vertically in water with its bounding diameter in the surface, is $41.6\,r^3$. Show that the depth of the center of pressure is $3\,\pi r/16$.

31. Show that if a triangle is immersed in a liquid with its plane vertical and one side in the surface, the center of pressure is at the middle of the median drawn to the lowest vertex.

32. Show that if a triangle is immersed in a liquid with its plane vertical and one vertex in the surface, the opposite side being parallel to the surface, the center of pressure divides the median drawn from the highest vertex in the ratio 3 : 1.

33. Calculate the mean ordinate of one arch of a sine-curve. The mean square ordinate. [Effective E. M. F. in an alternating electric current.]

34. Calculate the average distance of the points of a square from one corner.

35. What is the average distance of the points of a semicircular arc from the bounding diameter?

36. When a liquid flows through a pipe of radius R, the speed of flow at a distance r from the center is proportional to $R^2 - r^2$. What is the average speed over a cross section? What is the quantity of flow per unit time across any section?

37. The kinetic energy E of a moving mass is $\lim \sum \Delta m \cdot v^2/2$, where Δm is the element of mass moving with speed v. Show that for a disk rotating with angular speed ω, $E = \omega^2 I/2$. Calculate E for a solid car wheel of steel, 30 in. in diameter and 4 in. thick when the car is going 20 m./hr.

38. Show that the kinetic energy E of a sphere rotating about a diameter with angular speed ω is $(1/5)(\text{mass})r^2\omega^2$.

39. Calculate the kinetic energy in foot-pounds of the rim of a flywheel whose inner diameter is 3 ft., cross section a square 6 in. on a side, if its angular speed is 100 R. P. M. and its density is 7.

40. The x-component of the attraction between two particles m and m', separated by a distance r, is $(k \cdot m \cdot m'/r^2)\cos(r, x)$ where $\cos(r, x)$ denotes the cosine of the angle between r and the x-axis. Hence the x-component of the attraction between two elementary parts of two solids M and M' is $(k \cdot \Delta M \cdot \Delta M'/r^2)\cos(r, x)$. Show that the total attraction between the two solids is expressible by a six-fold integral.

41. A uniform rod attracts an external particle m. Calculate the components of the attraction parallel and perpendicular to the rod; the resultant attraction and its direction.

[Hint. Let ΔM be an element of the rod; then $\Delta F = k\Delta M \cdot m/r^2$ is the force due to ΔM acting on m, r being the distance from ΔM to m; then the components of ΔF are $\Delta X = \Delta F \cos\alpha$ and $\Delta Y = \Delta F \sin\alpha$, where α is the angle between r and the rod. Hence

$$X = \int \frac{kmdM}{r^2}\cos\alpha, \text{ and } Y = \int \frac{kmdM}{r^2}\sin\alpha.]$$

42. A force at O attracts a particle at P proportionally to the nth power of the distance OP. What is the average force from P_1 to P_2?

CHAPTER VIII

METHODS OF APPROXIMATION

PART I. EMPIRICAL CURVES INCREMENTS INTEGRATING DEVICES

119. Empirical Curves. Some of the methods used in science to draw the curves which represent simultaneous values of two related quantities and to obtain an equation which represents that relation approximately are given in Analytic Geometry. Usually the pairs of corresponding values are plotted on squared paper first; in all that follows it is assumed that this has been done in each case.

120. Polynomial Approximations. It is advantageous to have equations which are as simple as possible. From experimental results, it is not to be expected that absolutely precise equations can be found, and the attempt is made to get an equation of simple form which *approximately* represents the facts, in so far as the facts themselves are known. One simple kind of function which often does approximately express the facts is a **polynomial**:

(1) $$y = a + bx + cx^2 + dx^3 + \cdots + kx^n.$$

121. Review of Elementary Methods. If the points lie reasonably close to some straight line, it is usual to assume $n = 1$ in (1), § 120, whence $y = a + bx$; then b (the slope) and a (the y-intercept) may be found by direct measurements in the figure, or by one of the more general methods which follow.

If the curve has the typical form of a parabola, it is advantageous to assume that the equation is of the form

$$(2a)\ y = a + bx + cx^2, \quad \text{or} \quad (2b)\ (y - B) = C(x - A)^2$$

and then apply the methods of Analytic Geometry to find a, b, c, or A, B, C. One of the methods most often used is to find a, b, c, by assuming that the curve actually passes through three given points (see Ex. 2, p. 230).

Another method that can be be used whenever the vertex of the parabola is clearly indicated, is based on the fact that (A, B) are precisely the coördinates of the vertex, and can therefore be measured directly. The value of C, which is all that remains to be found, can be obtained approximately by a variety of methods: one may lay over the experimental figure a sheet of transparent (tracing) paper on which the curves $y = kx^2$ have been drawn for a large number of values of k; or one may proceed as in § 122; or, finally, as in § 124, below.*

In general, the equation (1) contains $n + 1$ unknown coefficients. To obtain these values, it is possible to use any $n + 1$ points on the experimental curve, as in Analytic Geometry. In doing so, it is preferable to take, not the precise figures given by the experiment, but rather pairs of coördinates of points on a free-hand curve sketched into the figure.

General formulas for the values of the coefficients have been worked out, and are given in the *Tables*, II, I, 17, under the name *Lagrange's Interpolation Formula.*

In the theory of probabilities, formulas are derived (which are to be found in any large set of mathematical tables) for the most *probable* values of the coefficients a, b, c, d, etc. These formulas can be applied by any person even before studying the theory. See *Tables*, II, D, 4.

* In any method, **judgment** on the part of the experimenter is the final means of deciding whether the equation obtained will approximately represent the facts. The amount of error which may exist in the experimental measurements is, of course, fundamentally important.

A few simple problems have been solved already by one of the methods of probabilities: in Exs. 18–23, p. 69, we assumed a formula of the type $y = kx$, and found k by the requirement that the sum of the squares of the errors should be a minimum. This method is called the **method of least squares**; see also Example 2, § 165.

122. Logarithmic Plotting. The preceding forms of equations may not represent the facts very well unless a large number of terms of (1), § 120, are used.

If the first graph resembles one of the curves $y = x^2$, $y = x^3$, $y = x^4$, etc., or $y = x^{1/2}$, $y = x^{1/3}$, etc., or $y = 1/x$, $y = 1/x^2$, etc., it is advantageous to plot the **common logarithms of the quantities measured** instead of the actual values of those quantities.

If x and y represent the quantities measured, and $u = \log_{10} x$, $v = \log_{10} y$ are their common logarithms, the values of u and v may lie very nearly on a straight line,

$$v = a + bu, \tag{1}$$

where a and b are found as in § 121. Then from (1), since $u = \log_{10} x$, $v = \log_{10} y$,

$$\log_{10} y = a + b \log_{10} x = \log_{10} k + \log_{10} x^b = \log_{10} (kx^b), \tag{2}$$

where $\log_{10} k = a$; hence

$$y = kx^b. \tag{3}$$

This form of equation is very convenient for computation and is used in practice very extensively wherever the logarithmic graph is approximately a straight line.* This work applies equally well for negative and fractional values of b.

* To avoid the trouble of looking up the logarithms, a special paper usually described in Analytic Geometry may be purchased which is ruled with logarithmic intervals. No particular explanation of this paper is necessary except to say that it is so made that if the values of x and y are plotted directly, the graph is identical with that described above. To secure this result the

In many cases where the process just described fails, it is sometimes advantageous to assume that the equation has the form $(y-B)=k(x-A)^n$ which evidently has a horizontal tangent at the point (A, B) if $n>1$, or a vertical tangent if $n<1$. If the first graph (in x and y) shows such a vertical or horizontal tangent, that point (A, B) may be selected as a new origin, and the values $x'=x-A$ and $y'=x-B$ should be used; thus we would plot the values of

$$u=\log_{10}x'=\log_{10}(x-A),\quad v=\log_{10}y'=\log_{10}(y-B),$$

in the manner described above. The values of A and B are found from the first graph (in x and y); the values of k and n are found from the logarithmic graph as above.

123. Semi-logarithmic Plotting. Variations of this process of § 122 are described in Exercises XLIX below. In particular, if the quantities are supposed to follow a *compound interest law*, $y=ke^{bx}$, it is advantageous to take logarithms of both sides:

$$\log_{10}y=\log_{10}k+bx\log_{10}e,$$

and then plot $u=x$, $v=\log_{10}y$; if the facts are approximately represented by any compound interest law, the experimental graph (in u and v) should coincide (approximately) with the straight line

$$v=A+Bu,$$

where $A=\log_{10}k$ and $B=b\log_{10}e$. After A and B have been measured, k and $b\,[=B\log_e 10=2.303\,B]$ can be found.

EXERCISES XLIX. — EMPIRICAL CURVES: ELEMENTARY METHODS

1. Find the equation of a straight line through the points $(-1, 3)$ and $(2, 5)$; through $(2, -3)$ and $(4, 5)$.

2. Determine a parabola whose axis is vertical, through the three points $(0, 3)$, $(2, -1)$, $(5, 8)$.

[HINT: Assume the equation in each of the forms $y=ax^2+bx+c$, $y-B=C(x-A)^2$; check the answers by comparing them.]

successive rulings are drawn at distances proportional to $\log 1 (=0)$, $\log 2$, $\log 3$, $\cdots$ from one corner, both horizontally and vertically.

Explanations and numerous figures are to be found in many books; see, *e.g.*, Kent, "Mechanical Engineers' Pocket Book" (Wiley, 1910), p. 85; Trautwine, "Civil Engineers' Pocket Book" (Wiley), (Chapter on Hydraulics).

3. Determine a cubic function of x which takes on the values -10, -2, 6, 20, respectively, when $x = -1, 0, 1, 2$.

4. Determine n and c so that the curve $y = cx^n$ passes through the two points $(1, 2)$ and $(3, 54)$; through $(1, 3)$, $(4, 6)$; through $(1, 3)$, $(8, 12)$.

5. Plot the data of Ex. 18, List XIV, p. 69; draw a straight line as closely as possible through the points without giving a preference to any one; determine the equation from this graph; compare it with the result obtained in List XIV.

6. Proceed as in Ex. 5 for each of the cases in Exs. 19–23, List XIV.

7. Assuming the data of Ex. 1, § 124, p. 235, find graphically the equation connecting f and w and compare it with the result found in § 124.

8. Assuming the data of Ex. 2, § 124, sketch a parabola whose axis is parallel to the axis of θ; determine its equation; compare the result with that of § 124.

9. Find a parabolic curve of the second degree which coincides with $y = \sin x$ at the points where $x = 0$, $x = \pi/2$, $x = \pi$. Compare the areas under the two curves.

10. Proceed as in Ex. 9 for each of the following curves, taking the values of x specified in each case:

(a) $y = \log_{10} x$,	$x = 1$,	$x = 5$,	$x = 10$.
(b) $y = e^x$,	$x = -1$,	$x = 0$,	$x = +1$.
(c) $y = \tan x$,	$x = 0$,	$x = \pi/3$,	$x = \pi/6$.
(d) $y = x^3 - 7x + 2$,	$x = 0$,	$x = 2$,	$x = 4$.

11. Find a parabolic curve of the third degree through four points taken at equal horizontal intervals on the curve $y = \sin x$, between $x = 0$ and $x = \pi/2$. Compare the areas under the two curves.

12. Find a parabolic curve of the second degree which coincides with $y = \sin x$ at $x = 0$ and $x = \pi/2$, and which has the same slope as $y = \sin x$ at $x = 0$.

13. Find a polynomial of second degree which, together with its first and second derivative, coincides with $\cos x$ at $x = 0$.

14. Proceed as in Ex. 12 for the curve $y = e^x$.

15. Find a cubic which, together with its first three derivatives, coincides with each of the following functions when $x = 0$:

(a) $\sin x$, (b) $\tan x$, (c) e^x, (d) $1/(1 + x)$.

16. Plot each of the following curves logarithmically, — either by plotting $\log_{10} x$ and $\log_{10} y$, or else by using logarithmic paper:

(*a*) $y = 2x^3$. (*c*) $y = .4x^{3.2}$. (*e*) $y = 5.7x^{.6}$.

(*b*) $y = 3x^{1/2}$. (*d*) $y = 3x^{-2}$. (*f*) $y = -1.4x^{2.4}$.

17. In each of the following tables, the quantities are the results of actual experiments; the two variables are supposed theoretically to be connected by an equation of the form $y = kx^n$. Draw a logarithmic graph and determine k and n, approximately:

(*a*) [Steam pressure; v = volume, p = pressure.] [Saxelby].

v	2	4	6	8	10
p	68.7	31.3	19.8	14.3	11.3

(*b*) [Gas engine mixture; notation as above.] [Gibson.]

v	3.54	4.13	4.73	5.35	5.94	6.55	7.14	7.73	8.04
p	141.3	115	95	81.4	71.2	63.5	54.6	50.7	45

(*c*) [Head of water h, and time t of discharge of a given amount.] [Gibson.]

h	0.043	0.057	0.077	0.095	0.100
t	1260	540	275	170	138

(*d*) [Heat conduction, asbestos; θ = temperature (F.), C = coefficient of conductivity.] [Kent.]

θ	32°	212°	392°	572°	752°	1112°
C	1.048	1.346	1.451	1.499	1.548	1.644

(e) [Track records: d = distance, t = record time (intercollegiate).]

d	100 yd.	220 yd.	440 yd.	880 yd.	1 mi.	2 mi.
t	$0:09\frac{4}{5}$	$0:21\frac{1}{5}$	$0:48\frac{4}{5}$	$1:56$	$4:17\frac{1}{5}$	$9:27\frac{3}{5}$

[NOTE. See *Kennelly*, Fatigue, etc., *Proc. Amer. Acad. Sc.* XLII, No. 15, Dec. 1906; and *Popular Science Monthly*, Nov. 1908.]

18. Plot the following curves, using logarithmic values of one quantity and natural values of the other:

(a) $y = e^x$. (b) $y = 10\,e^{3x}$. (c) $y = 4\,e^{-x}$. (d) $y = .1\,e^{-x/3}$.

19. Discover a formula of the type $y = ke^{ax}$ for each of the following sets of data:

(a)	x:	.2	.4	.6	.8	1.0
	y:	1.5	2.2	3.3	5.0	7.4
(b)	x:	.6	1.2	1.8	2.4	3.0
	y:	3.0	4.4	6.6	10.0	14.8
(c)	x:	.31	.63	.94	1.26	1.57
	y:	1.22	1.49	1.82	2.23	2.72
(d)	x:	.2	.8	2.0	4.0	
	y:	.82	.45	.13	.02.	
(e)	x:	.63	1.26	2.51	3.77	5.03
	y:	2.01	1.35	.60	.27	.12
(f)	x:	1	2	3	4	5
	y:	1.63	1.34	1.08	.90	.73

20. A is the amplitude of vibration of a long pendulum, t is the time since it was set swinging. Show that they are connected by a law of the form $A = ke^{-nt}$.

A in. =	10	4.97	2.47	1.22	.61	.30	.14
t min. =	0	1	2	3	4	5	6

124. Method of Increments. A method which is often better in practice than those in § 121 is as follows. If the curve is supposed to be a parabola,

$$y = a + bx + cx^2, \tag{1}$$

and if we take two pairs of values of x and y, say (x, y) and $(x+\Delta x, y+\Delta y)$ given by experiment, we should have

(2) $\quad y = a + bx + cx^2, \quad y + \Delta y = a + b(x+\Delta x) + c(x+\Delta x)^2,$

whence

(3) $$\Delta y = b\,\Delta x + 2\,cx\,\Delta x + c\,\overline{\Delta x}^2.$$

If Δx is constant, *i.e.* if points are selected at equal intervals on the crudely sketched curve drawn through the experimental points, we might write

(4) $$Y = \Delta y = (bh + ch^2) + 2\,ch \cdot x = A + Bx$$

where $h = \Delta x$. If we should actually plot this equation, $Y = A + Bx$, we would get (approximately) a straight line. Now $\Delta y = Y$ is the difference of two values of y; it can be found for each of the values of x selected above, and the (approximate) straight line can be drawn, so that A and B can be measured as in § 121.

We may repeat the preceding process; from (4) we obtain, as above,

(5) $$\Delta Y = B\,\Delta x = 2\,ch^2, \quad (h = \Delta x),$$

whence ΔY is *constant* if h was taken constant. Now ΔY is the difference between two values of Y; that is, ΔY is the difference between two values of Δy:

$$\Delta Y = \Delta(\Delta y) = \Delta^2 y,$$

and for that reason is called a **second difference**, or a **second increment.** If the second differences are reasonably constant, we conclude that an equation of the form (1) will reasonably represent the facts and we find c directly by solving equation (5).

Example 1. With a certain crane it is found that the forces f measured in pounds which will just overcome a weight w are

f	8.5	12.8	17.0	21.4	25.6	29.9	34.2	38.5
w	100	200	300	400	500	600	700	800

What is the law connecting power with the weight that it just overcomes? [PERRY.]

Plotting the values of f and w, it appears that the points are very nearly on a straight line $f = a + bw$. If they were on a straight line, $\Delta f/\Delta w$ would be constant and equal to $df/dw = b$. As a matter of fact, for each increase of weight, $\Delta f/\Delta w$ varies only from .042 to .044, its average value being $30/700 = .0429$. Taking this value for b, one gets for the equation of the line, and hence for the relation between power and weight:

$$f = 4.21 + .0429\,w, \quad 4.21 = 8.5 - 100 \times .0429.$$

Here 4.21 appears to be the power needed to start the crane if no load were to be lifted.

Example 2. If θ is the melting point (Centigrade) of an alloy of lead and zinc containing $x\,\%$ of lead, it is found that

$x = \%$ lead	40	50	60	70	80	90
$\theta =$ melting point	186	205	226	250	276	304

Plotting the points (x, θ) will show them not to lie in a straight line as is also shown by the differences $\Delta\theta$. But $\Delta(\Delta\theta)$ or $\Delta^2\theta$ does run uniformly. Therefore one tries a quadratic function of x for θ, that is

$$\theta = a + bx + cx^2.$$

It is evident that $$\Delta\theta = 10\,b + c\,(20\,x + 100),$$

and $$\Delta^2\theta = 200\,c.$$

The average value of $\Delta^2\theta$ is 2.25. Hence $c = .01125$. If we subtract cx^2 from θ, we find $\theta - cx^2 = a + bx$. These values can be calculated from the data and from $c = .01125$; they will be found to lie on a straight line;

hence a and b can be found by any one of several preceding methods. The student will readily obtain, approximately,

$$\theta = 133 + .875\,x + .01125\,x^2,$$

a formula which represents reasonably the melting point of any zinc-lead alloy. [SAXELBY.]

EXERCISES L.—EMPIRICAL CURVES BY INCREMENTS

1. Express $f(x)$ as a quadratic function of x, when

x:	0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$:	2.5	1.9	1.6	1.5	1.7	2.1	2.8.

2. Express $f(x)$ as a cubic function of x, when

x:	0	.02	.04	.06	.08	.10	.12	.14
$f(x)$:	0	.020	.042	.064	.087	.111	.136	.163.

3. Express $\phi(m)$ as a cubic in m, when

m:	.01	.02	.03	.04	.05	.06	.07	.08
$\phi(m)$:	.00010	.00041	.00093	.00166	.00260	.00385	.00530	.00690.

4. The specific heat S of water, at $\theta°$ C., is

θ:	0	5	10	15	20	25	30
S:	1.0066	1.0038	1.0015	1.0000	0.9995	1.0000	1.002.

Express S in terms of θ.

5. Determine a relation between the vapor pressure P of mercury, and the temperature θ C., from the data below:

θ:	60	90	120	150	180	210	240
P:	.03	.16	.78	2.93	9.23	25.12	58.8.

6. The resistance R, in ohms per 1000 feet, of copper wire of diameter D mils, is

D:	289	182	102	57	32	18	10
R:	.126	.317	1.010	3.234	10.26	32.8	105.1.

Find a relation between R and D.

7. The Brown and Sharpe gauge numbers N of wire of diameter D mils, are

N:	1	5	10	15	20	25	30
D:	289	182	102	57	32	18	10.

Express D in terms of N.

8. Find a relation between the speed S of a train in kilometers per hour, and the horse-power (H. P.) of the engine from the data below:

H. P.:	550	650	750	850
S:	26	35	52	70.

9. Determine a relation between the age of a lamp and its candle power (C. P.) from the following data:

Hours:	0	250	500	750	1000	1250	1500
C. P.:	24.0	17.6	16.5	15.8	15.3	14.9	14.5.

10. Proceed as in Ex. 9 for each of the two lamps (I and II) below:

Hours:		0	250	500	750	1250	1750	2250	2750
C. P.	I:	13.70	15.80	16.65	16.50	14.50	13.25	12.00	11.40
	II:	17.75	20.00	19.00	18.60	17.90	17.00	15.50	14.10.

11. The dip, θ, of the magnetic needle at Harrisburg, Pa., was observed as below:

Date:	1840.5	1862.6	1877.7	1885.6	1895.7
Dip:	72°.34	72°.50	72°.34	71°.75	71°.72.

Show that $\theta = 72^\circ.48 + ^\circ.0067\, m - ^\circ.00056\, m^2$, where $m =$ date $- 1850$. At what date was the dip greatest?

12. Proceed as in Ex. 11 for the data below, taken at Eastport, Me.:

Date:	1860.5	1863.5	1865.6	1873.7	1879.6	1887.6	1895.6
Dip:	75°.88	75°.80	75°.74	75°.41	75°.20	74°.90	74°.63.

Show that $\theta = 76^\circ.31 - 0^\circ.039\, m + ^\circ.000053\, m^2$, where $m =$ date $- 1850$.

13. The intensity of illumination at the same distance but in different directions from an incandescent lamp was observed as below, $\theta = 0^\circ$ being downward and $\theta = 90^\circ$ horizontally from the lamp:

θ:	0	30	60	90	120	150
C. P.:	6.6	9.5	14.5	16.0	14.5	9.8.

Lay off C. P. and θ in rectangular and also in polar coördinates, and find a relation between them.

14. Proceed as in Ex. 13 for a shaded lamp:

θ:	0	10	20	30	40	50	60	70
C. P.:	47.3	44.2	36.6	30.0	25.6	22.0	17.6	10.6.

15. The energy consumed in overcoming molecular friction when iron is magnetized and demagnetized (hysteresis, H, — measured in watts per cycle per liter of iron) is given below in terms of the strength of the magnetic field (B, — measured in lines per square centimeter). What is the relation between them ?

B:	2000	4000	6000	8000	10000	14000	16000	18000
H:	.022	.048	.085	.138	.185	.320	.400	.475.

16. Proceed as in Ex. 15, for cobalt, the hysteresis loss H being now measured in ergs per cycle per second :

B:	900	2350	3100	4100	4600	5200	5850	6500
H:	450	2450	3950	6300	7400	8950	10950	13250.

17. The table below contains some data on the comparison of a tungsten lamp with a tantalum lamp. The voltage or electrical pressure V, is in volts, the resistance R, in ohms, the current consumed in watts per candle power ; C denotes candle power, and W watts per candle power.

	Tungsten			Tantalum		
Voltage V	C. P. C	Watts per C. P. W	Resistance R	C. P. C	Watts per C. P. W	Resistance R
80	14	2.51	166	5	3.80	260
90	24	1.83	173	10	2.85	265
100	36	1.49	182	18	2.05	275
110	52	1.23	190	25	1.65	283
120	71	1.10	197	38	1.35	290
130	95	0.96	202	50	1.15	300
140	128	0.83	210	62	0.95	308
150	160	0.76	216	78	0.85	315
160	196	0.58	222	100	0.75	323
170	230	0.52	227	122	0.70	327
180	270	0.50	232	156	0.70	332
190	312	0.48	238	190	0.60	340
200	340	0.47	242	235	0.55	345

For each lamp, express each of the quantities C, W, R, in terms of V.

125. Approximate Integration. One method of finding approximate values of a definite integral is that used in defining an integral, § 66, p. 114. This consists in finding special values of the sums S and s of p. 114, by breaking up the interval between the limits into many parts, and combining portions approximately as if they were rectangles:

$$S = \Sigma f(x_R)\,\Delta x, \quad s = \Sigma f(x_L)\,\Delta x, \tag{1}$$

where x_R and x_L denote, respectively, the values of x at the right and left ends of the partial interval.

A still better value is obtained by averaging these two:

$$\frac{S+s}{2} = \sum \frac{f(x_R)+f(x_L)}{2}\,\Delta x, \tag{2}$$

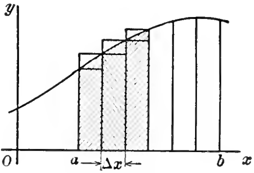


FIG. 52

for this amounts to the same thing as replacing each partial area by the **trapezoid** whose base is Δx and whose sides are $f(x_R)$ and $f(x_L)$. See footnote, p. 112.

Finally the **prismoid rule** (§ 71, and Ex. 10, p. 129) gives

$$\int_{x=a}^{x=b} f(x)\,dx = \frac{f(a) + 4f\left(\dfrac{a+b}{2}\right) + f(b)}{6}(b-a), \tag{3}$$

which amounts to replacing the curve by a parabolic arc $y = Ax^2 + Bx + C$ through its end points ($x = a$ and $x = b$) and its middle point $x = (a+b)/2$.

If the prismoid rule is applied to each successive pair of an even number of subdivisions of width Δx each, and if $x = x_1, x_2, x_3, \cdots, x_{n-1}$ be the values of x at the division

points, we find, approximately,

$$(4)\quad \int_{x=a}^{x=b} f(x)\,dx$$

$$=\frac{f(a)+4f(x_1)+f(x_2)}{6}(2\,\Delta x)+\frac{f(x_2)+4f(x_3)+f(x_4)}{6}(2\,\Delta x)+\cdots$$

$$=\frac{f(a)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+f(b)}{3}\cdot\Delta x$$

which is known as **Simpson's Rule.** See Ex. 12, p. 129.

All these rules evidently apply to the approximate *computation* of any integral, no matter where it arose.

126. Integration from Empirical Formulas. Limit of Error. If a formula $y=f(x)$ has been obtained empirically, it may be used to find the area under the curve represented by the experimental data. If the **maximum error** due to experimental errors and to faulty approximation is M so that the true value of y differs from $f(x)$ by at most M, we have,* if $b>a$,

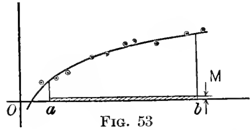


FIG. 53

$$\left|\int_{x=a}^{x=b} y\,dx\right| \leqq \left|\int_{x=a}^{x=b} f(x)\,dx\right| + \left|\int_{x=a}^{x=b} M\,dx\right|$$

$$\leqq \left|\int_{x=a}^{x=b} f(x)\,dx\right| + M(b-a);$$

that is, the error in the value of the integral calculated by using the approximation formula $y=f(x)$ is not greater than $M(b-a)$.

* The pair of vertical lines | | indicate, as before (see pp. 16, 171), the *positive numerical value* (or *absolute* value) of the quantity inclosed.

The same result applies in cases in which a function to be integrated has been replaced, for convenience, by a simpler function.

Thus $$\frac{1}{1-x} = 1 + x + x^2 + \frac{x^3}{1-x}.$$

If we replace $1/(1-x)$, for convenience, by $1 + x + x^2$, the error E made in doing so is:

$$E = \frac{x^3}{1-x}$$

which, for values of x numerically less than $1/10$, is numerically less than $(.1)^3/.9 < .0012$; hence if we write

$$\int_{x=0}^{x=.1} \frac{1}{1-x}\,dx = \int_{x=0}^{x=.1} (1 + x + x^2)\,dx = \left[x + \frac{x^2}{2} + \frac{x^3}{3}\right]_{x=0}^{x=.1} = .10533,$$

the error E made in the value of the integral is *less* than $.1 \cdot .0012 = .00012$. The exact value of the original integral is
$-\log(1-x)\Big]_{x=0}^{x=.1} = -\log(.9) = -\log_{10}(.9)\log_e 10 = .045757 \cdot 2.30258 =$ $.10536$. In general, as in the example, the final error may be very much less than the estimated upper limit of the error calculated above.

127. Derived and Integral Curves. In § 49, p. 77, we drew the derived curves by finding the derivatives and plotting their values.

If the original curve was drawn from values found by some experiment, and if its equation is unknown, the derived curve can be drawn mechanically. To do so, draw, according to your best judgment, the tangents at each of a large number of points (x_0, y_0), (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n), noting about how much uncertainty there seems to be in each case. Find the slope m_i of the tangent at each point (x_i, y_i) by measuring its rise per horizontal unit. Plot the points (m_i, x_i), indicating the estimated uncertainty in each value of m. Draw a smooth curve which passes near each of these points, allowing the most variation at the points where the values of m seemed to be most uncertain. Check by comparing the slope of the original curve and the ordinate of the derived curve for various other values of x. This process may not be very reliable, and every possible check must be used. (See § 143 (d).)

Likewise, if any function $y = f(x)$ is given, the **integral curve**:

$$I = \int f(x)\,dx = \phi(x) + C,$$

which represents the area under $y = f(x)$ from some fixed left-hand boundary to the ordinate $x = x$ can be drawn.* But if the equation of the curve is not known, this can still be done by the methods of § 125; or by simply estimating the area from some left-hand vertical line up to various points x_1, x_2, $\cdots$, x_n and marking at each value of x, as a new ordinate, the value of the area up to that point. The result is surprisingly accurate if the curve is drawn on millimeter paper and the area obtained by actually counting the squares. The accuracy of this process as compared with the uncertainty of mechanical construction of the derived curves, is a consequence of § 126.

EXERCISES LI. — APPROXIMATE EVALUATION OF INTEGRALS

1. Find the area under the curve $y = 1/(1 - x)$ from $x = 0$ to $x = .1$ by use of the prismoid formula, and show that the result is accurate to five decimal places.

2. Draw the curve $y = 1/(1 - x)$ and construct the integral curve from $x = 0$ to any value of x less than 1, first by actually counting the squares on the cross section paper, second by actually integrating between the limits $x = 0$ and $x = x$.

3. Find the area under the curve $y = 1/x^2$ between $x = 1$ and $x = 2$, approximately, first by using the prismoid formula, then by using Simpson's rule with three intermediate points of division. Compare the results with the precise answer obtained by integration.

4. Find the error made in computing the value of the area of one arch of the curve $y = \sin x$ if the approximating parabola of Ex. 9, List XLIX, p. 231, is used instead of the sine curve.

5. Proceed as in Ex. 4, for each of the curves and their approximating parabolas mentioned in Ex. 10, List XLIX, taking the extreme values of x mentioned there as limits of integration.

* Different values of C give, of course, different integral curves, all congruent, obtained from any one of them by a stiff vertical motion.

6. Show that $x^{2.5}$ lies between x^2 and x^3 from $x = 0$ to $x = 1$. Hence show that $\int_0^1 x^{2.5}\,dx$ lies between 1/3 and 1/4. Find the exact value of the integral.

7. Show that $1/\sqrt{1-x^4}$ lies between $1/\sqrt{1-x^2}$ and $1/\sqrt{2(1-x^2)}$ between $x = 0$ and $x = 1$; hence find extreme limits between which $\int_0^1 [1/\sqrt{1-x^4}]\,dx$ lies.

8. Compute the value of the integral $\int_0^1 [1/(1+x^2)]\,dx$ (*a*) by the prismoid rule; (*b*) by the trapezoid rule, with two intermediate points of division; (*c*) by Simpson's rule, with three intermediate points of division; (*d*) precisely by direct integration. Compare the results for accuracy.

9. Show by long division that $1/(1+x^2) = 1 - x^2 + x^4 - x^6/(1+x^2)$. Hence show that the area under $y = 1/(1+x^2)$ from $x = 0$ to $x = .5$ differs from that under $y = 1 - x^2 + x^4$ by less than 1/128. Actually compute both areas, and show that this estimate of the error is far larger than the actual error.

10. Draw the curve $y = 1/(1+x^2)$ and construct its integral curve, starting from the initial point $x = 0$. Verify by direct integration.

11. Draw the curve $y = e^{x-2}$ and construct its integral curve. Find the value of the integral from $x = 0$ to $x = 1$, approximately, (*a*) from this integral curve; (*b*) by the prismoid formula; (*c*) by the trapezoid rule, with one intermediate point; (*d*) by Simpson's rule, with one intermediate point.

12. In each of the exercises of Ex. 17, List XLIX, p. 232, estimate from the figure an upper limit of the difference between the given data and the values represented by the empirical formula obtained. Hence find an upper limit of the total error which would be made in using the empirical formula to find the area underneath the curve.

13. Given a function $f(x)$ defined by the following set of data:

x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
$f(x)$	1	.9	.7	.4	0	$-.4$	$-.7$	$-.9$	$-1.$	$-.9$	-1

find approximately the derivative of $f(x)$ at each of the points $x = .2$, $x = .3$, $x = .7$. Find approximately the value of the integral of $f(x)$ from $x = 0$ to each of the preceding values of x.

128. Integrating Devices. It is important in many practical cases to know approximately the areas of given closed curves. Thus the volume of a ship is found by finding the areas of cross sections at small intervals.

Besides the methods described above, the following devices are employed:

A. Counting squares on cross section paper.

B. Weighing the figures cut from a heavy cardboard of uniform known weight per square inch.

C. *Integraphs.* These are machines which draw the integral curve mechanically; from it values of the area may be read off as heights.

The simplest such machine is that invented by ABDANK-ABAKANOWICZ. A heavy carriage $CDEF$ on large rough rollers, R, R' is placed on the paper so that CE is parallel to the y-axis.

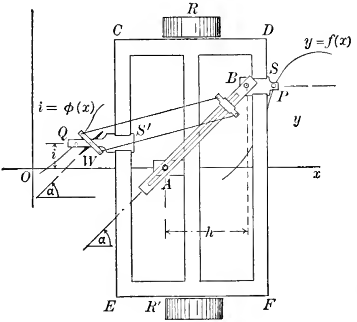


FIG. 54

Two sliders S and S' move on the parallel sides DF and CE; to S is attached a pointer P which follows the curve $y = f(x)$. A grooved rod AB slides over a pivot at A, which lies on the x-axis, and is fastened by pivot B to the slider S. A parallelogram mechanism forces a sharp wheel W attached to the slider S' to remain parallel to AB. A marker Q draws a new curve $i = \phi(x)$, which obviously has a tangent parallel to W, that is, to AB. If AB makes an angle α with Ox, $\tan\alpha$ is the slope of the new curve; but $\tan\alpha$ is the height of S divided by the fixed horizontal distance h between A and B:

$$\frac{d\,\phi(x)}{dx} = \tan\alpha = \frac{\text{height of } S}{h} = \frac{f(x)}{h};$$

whence

$$i - i_0 = \frac{1}{h}\int_{x=a}^{x=x} f(x)\,dx;$$

where a is the value of x at P when the machine starts, and i_0 denotes the vertical height of the new curve at the corresponding point.

D. Polar Planimeters. — There are machines which read off the area directly (for any smooth closed curve of simple shape) on a dial attached to a rolling wheel.

The simplest such machine is that invented by AMSLER.

Let us first suppose that a moving rod ab of length l always remains perpendicular to the path described by its center C. The path of C may be regarded as the limit of an inscribed polygon, and the area swept over by the rod may be thought of as the limit of the sum of small quadrilaterals, the area ΔA of each of which is $l\Delta p$, approximately, where Δp is the length of the corresponding side of the polygon inscribed in the path of C. Hence the total area A swept over by the rod is evidently lp, where p is the total length of the path of C.

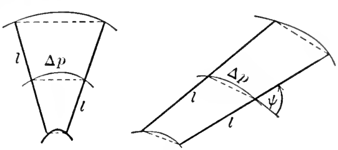


FIG. 55

But if the rod does not remain perpendicular to the path of C during the motion, and if ψ is the angle between the rod and that path, the area ΔA becomes $l \sin \psi \cdot \Delta p$, approximately. The expression $\sin \psi \cdot \Delta p$ may be thought of as the component of Δp in a direction perpendicular to the rod. Calling this component Δs, we have $\Delta A = l\Delta s$, approximately; and the total area A swept over by the rod is precisely $\lim \Sigma \Delta A = \lim \Sigma l \Delta s = \int l\, ds = l \int ds = ls$, where $s = \int ds$ is the total motion of C in a direction perpendicular to the rod.

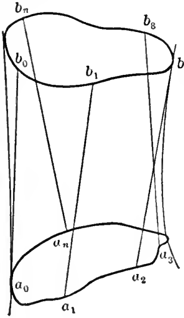


FIG. 56

The quantity $s = \int ds$ can be measured mechanically by means of a wheel of which the rod is the axle, attached to the rod at C; for if θ is the total angle through which the wheel turns during the motion, $s = r\theta$, where r is the radius of the wheel, and θ is measured in radians. Hence $A = ls = lr\theta$; the value of θ is read off from a dial attached to the wheel; l and r are known lengths.

In *Amsler's polar planimeter*, one end b of the rod ab is forced to trace once around a given closed curve whose area is desired; the other

end a is mechanically forced to move back and forth along a circular arc by being hinged at a to another rod Oa, which in its turn is hinged to a heavy metal block at O. As b describes that part of the given curve which lies farthest from O, the rod ab sweeps over an area between the circular arc traced by a and the outer part of the given curve; as b describes the part of the curve nearest to O, ab sweeps back over a portion of the area covered before, between the circle and the inner part of the given curve. This latter area does not count in the final total, since it has been swept over twice in opposite directions. Hence the quantity $A = lr\theta$, given by the reading of the dial on the machine, is precisely the desired area of the given closed curve, which has been swept over just once by the moving rod ab.

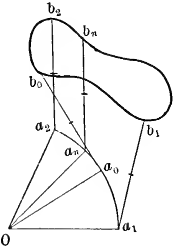


FIG. 57

In practicing with such a machine, begin with curves of known area. The machine is useful not only in finding areas of irregular curves whose equations are not known, but also in checking integrations performed by the standard methods, and in giving at least approximate values for integrals whose evaluation is difficult or impossible.

For further information on integrating devices, see: Abdank-Abakanowicz, *Les intégraphes* (Paris, Gauthier-Villars); Henrici, *Report on Planimeters* (British Assoc. 1894, pp. 496-523); Shaw, *Mechanical Integrators* (Proc. Inst. Civ. Engs. 1885, pp. 75-143); Encyclopädie der Math. Wiss., Vol. II. Catalogues of dealers in instruments also contain much really valuable information.

129. Tabulated Integral Values. — Another method of obtaining the values of certain integrals is to look them up in numerical tables which have been calculated by the foregoing methods or by other means. In these tables are printed the values of the integral I:

$$(1) \qquad I = \int_{x=a}^{x=u} f(x)\, dx,$$

where a is some convenient constant, for a large number of values of the (variable) upper limit u which differ by small amounts.

Thus tables of common logarithms are precisely tables of values of the integral

$$(2)\quad I=\int_{x=1}^{x=N}\frac{\log_{10}e}{x}\,dx=\log_{10}e\Big[\log_e x\Big]_{x=1}^{x=N}=\log_{10}x\Big]_{x=1}^{x=N}=\log_{10}N.$$

Among other integrals which may be found thus tabulated are the **inverse hyperbolic sine** (see *Tables*, V, C):

$$(3)\qquad I=\int_{x=0}^{x=u}\frac{dx}{\sqrt{x^2+1}}=\log\left(u+\sqrt{u^2+1}\right)=\sinh^{-1}u;$$

the **elliptic integral of the first kind** (see *Tables*, V, D):

$$(4)\quad I=\int_{x=0}^{x=u}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\int_{\alpha=0}^{\alpha=\phi}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}},$$

where $x=\sin\alpha$ and $u=\sin\phi$; and others which are defined in the *Tables*, V, E–H. Many other integrals can be reduced to these, just as many integrals can be expressed in terms of logarithms.

These tables give corresponding values of the integral I and its upper limit u; hence they define I as a function of u:

$$I=f(u);$$

that is, *the integral is a function of its* (variable) *upper limit*.

The values of these integrals can be read off also from a properly constructed graph in which their values are plotted in the usual manner. Thus the curve $u=\sinh I$ (see *Tables*, III, E) may be used to obtain, approximately, the value of I when u is given; that is, the values of the integral of equation (3) for given values of u.

EXERCISES LII. — INTEGRATING DEVICES NUMERICAL TABLES

1. Construct a figure of each of the types mentioned below, with dimensions selected at random, and find their areas approximately by counting squares; by Simpson's rule; by the planimeter, if one is available. (*a*) A right triangle. (*b*) An equilateral triangle. (*c*) A circle. (*d*) An ellipse. (Draw it with a thread and two pins.) (*e*) An arch of a sine curve. (*f*) An arch of a cycloid.

2. Evaluate the integrals below approximately, by drawing the graphs of the integrands.

(*a*) $\int_0^1 \sqrt{1+x^4}\,dx.$ (*d*) $\int_0^1 e^{x^2}\,dx.$ (*g*) $\int_0^{10} \sin e^{-x}\,dx.$

(*b*) $\int_0^{\sqrt{\pi}} \sin x^2\,dx.$ (*e*) $\int_0^{10} e^{-x^2}\,dx.$ (*h*) $\int_1^e \sqrt{\log x}\,dx.$

(*c*) $\int_0^{\pi^2} \sin \sqrt{x}\,dx.$ (*f*) $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx.$ (*i*) $\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx.$

3. The integral of y^2 from $x = a$ to $x = b$, divided by $b - a$, is called the *mean square ordinate* of the curve $y = f(x)$ from $x = a$ to $x = b$. Find the mean square ordinate for the curve $y = \sin x$, both by actual integration and by approximate methods. See Ex. 33, p. 226.

4. Show that the mean square ordinate of $y = k \sin x$ may be found graphically by plotting the circle $\rho = k \sin \theta$ in polar coördinates, since $k^2 \int_0^\pi \sin^2 \theta\, d\theta$ is twice the area of this circle.

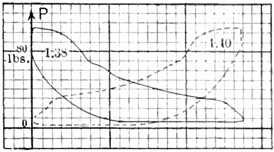
5. Show in general that the mean square ordinate of $y = f(x)$ can be found graphically from the polar figure for $\rho = f(\theta)$.

6. In the theory of electricity, it is shown that the effective electromotive force E of a current is the square root of the mean square of the actual (variable) electromotive force e. Use the method of Ex. 5 to find E if $e = 100 \sin \theta + 10 \sin 2\theta$, from $\theta = 0$ to $\theta = \pi$, where θ is the angle described by certain moving parts of the generating machinery.

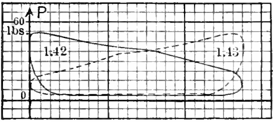
7. Proceed as in Ex. 6 for the following experimental values of e; from $\theta = 0$ to $\theta = 90°$:

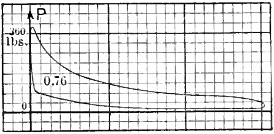
θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
e	3	50	64	52	58	107	130	150	165	145

8. The figures below are reproductions of indicator cards, taken from three different types of engines. The dotted curves are entirely separate from the full lines. The *average pressure* on the piston is the area of one of these curves divided by the length of stroke. Find this value in each case, where the stroke is 12 in. in the first figure, and 8 in. in each of the others. (Unit of area = 1 large square.)



[NOTE. The *work done* is precisely the area in question, on a proper scale, since the work is the average pressure times the length of stroke.]





9. A piece of land lies between a straight road and a river which crosses the road at two points. The perpendicular distances from road to river at intervals of 20 yd. are 0, 15, 35, 40, 50, 45, 45, 30, 20, 10, 5, 0 yd. Find approximately the area of the land by each of the methods described above.

10. Find from the tables the values of each of the following integrals:

(*a*) $\displaystyle\int_{1.5}^{2.1} \frac{dx}{x}$. (*b*) $\displaystyle\int_{0.2}^{2} \frac{e^x}{x}\,dx$. (*c*) $\displaystyle\frac{2}{\sqrt{\pi}}\int_{0.4}^{2.5} e^{-x^2}\,dx$.

(*d*) $\displaystyle\int_0^{\pi/3} \frac{d\theta}{\sqrt{1 - .09\sin^2\theta}}$. (*g*) $\displaystyle\int_0^{1/2} \frac{\sqrt{1 - .49\,x^2}}{\sqrt{1 - x^2}}\,dx$.

(*e*) $\displaystyle\int_0^{\pi/3} \sqrt{1 - .09\sin^2\theta}\,d\theta$. (*h*) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{d\theta}{\sqrt{1 - .64\sin^2\theta}}$.

(*f*) $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{(1 - x^2)(1 - .36\,x^2)}}$. (*i*) $\displaystyle\int_0^{\infty} e^{-x} x^{.6}\,dx$.

11. Reduce each of the following integrals to standard forms to be found in the tables by means of the substitutions indicated, and then evaluate them:

(*a*) $\displaystyle\int_{\pi/6}^{\pi/3} \sqrt{1 - .09 \cos^2 \theta}\, d\theta$; put $\theta = 90° - \psi$.

(*b*) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - .49 \sin^2 2\theta}}$; put $2\theta = \psi$.

(*c*) $\displaystyle\int_e^{e^2} \frac{dx}{\log x}$; put $\log x = u$.

(*d*) $\displaystyle\int_0^{1.6} e^{-\frac{x^2}{4}}\, dx$; put $\dfrac{x}{2} = u$.

(*e*) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(4 - x^2)(4 - .36\, x^2)}}$; put $\dfrac{x}{2} = u$.

PART II. POLYNOMIAL APPROXIMATIONS SERIES TAYLOR'S THEOREM

130. Rolle's Theorem. Let us consider a curve

$$y = f(x),$$

where $f(x)$ is single-valued and continuous, and where the curve has at every point a tangent that is not vertical. If such a curve cuts the x-axis twice, at $x = a$ and $x = b$, it surely either has a maximum or a minimum at at least one point $x = c$ between a and b. It was shown in § 39, p. 64, that the derivative at c is zero:

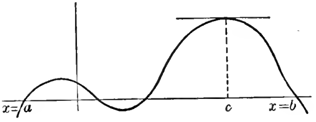


FIG. 58

$$[A] \quad \textit{If} \quad f(a) = f(b) = 0, \quad \textit{then} \left[\frac{df(x)}{dx}\right]_{x=c} = 0, \quad (a < c < b);$$

this fact is known as **Rolle's Theorem.**

131. The Law of the Mean. Rolle's Theorem is quite evident geometrically in the form: *An arc of a simple smooth curve cut off by the x-axis has at least one horizontal tangent.* The precise nature of the necessary restrictions is given in § 130.

Another similar statement, which is true under the same restrictions and is equally obvious geometrically, is: *An arc of a simple smooth curve cut off by any secant has at least one tangent parallel to that secant.*

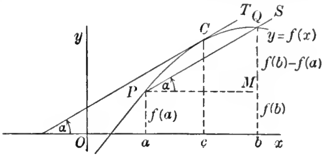


FIG. 59

If the curve is $y = f(x)$, and if the secant S cuts it at points P: $[a, f(a)]$ and Q: $[b, f(b)]$, the slope of S is

$$\Delta y \div \Delta x = [f(b) - f(a)] \div (b - a).$$

The slope of the tangent CT at $x = c$ is equal to this:

$$[B] \quad \left.\frac{dy}{dx}\right]_{x=c} = \left.\frac{df(x)}{dx}\right]_{x=c} = \frac{f(b) - f(a)}{b - a} = \frac{\Delta y}{\Delta x}, \qquad (a < c < b).$$

This statement is called the **Law of the Mean** or the **Theorem of Finite Differences.**

It is easy to prove this statement algebraically from Rolle's Theorem. For if we subtract the height of the secant S from the height of the curve, we get a new curve whose height is:

$$D(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

Now $D(x)$ is zero when $x = a$ and when $x = b$. It follows by § 130 that $dD(x)/dx = 0$ at $x = c$, $(a < c < b)$:

$$\left.\frac{dD(x)}{dx}\right]_{x=c} = \left(\frac{df(x)}{dx} - \left[\frac{f(b) - f(a)}{b - a}\right]\right)_{x=c} = 0, \qquad (a < c < b),$$

which is nothing but a restatement of $[B]$.

132. Increments. The law of the mean is used to determine increments approximately, and to evaluate small errors.

If $y = f(x)$ is a given function, we have, by § 131,

$$[B] \qquad \Delta y = \left.\frac{dy}{dx}\right]_{x=c} \cdot \Delta x.$$

In practice this law is used to estimate the extreme limit of errors, that is, the extreme limit of the numerical value of Δy. It is evident that

$$[B^*] \qquad |\Delta y| \leqq M_1 \cdot |\Delta x|,$$

where M_1 is the maximum of the numerical value of dy/dx between a and $a + \Delta x$. When Δx is very small, the slope dy/dx is practically constant from a to $a + \Delta x$ in most instances, and M_1 is practically the same as the value of dy/dx at any point between a and $a + \Delta x$.

Example 1. To find the correct increments in a five-place table of logarithms.

The usual logarithm table contains values of $L = \log_{10} N$ at intervals of size $\Delta N = .001$. Hence

$$\Delta L = \left[\frac{d \log_{10} N}{dN}\right]_{N=c} (.001) = \left[\frac{.001}{N} \log_{10} e\right]_{N=c} = \left[\frac{.00043}{N}\right]_{N=c},$$

where $N < c < N + .001$.

Logarithms are ordinarily given from $N = 1$ to $N = 10$. Hence ΔL will vary from .00043 at the beginning of the table to .00004 at the end of the table. This agrees with the "differences" column in an ordinary logarithm table.

Example 2. The reading of a certain galvanometer is proportional to the tangent of the angle through which the magnetic needle swings. Find the effect of an error in reading the angle on the computed value of the electric current measured. We have

$$C = k \tan \theta,$$

where C is the current and θ the angle reading. Hence the error E_C in the computed current is

$$E_C = \Delta C = \left.\frac{kd \tan \theta}{d\theta}\right]_{\theta=\alpha} \Delta\theta = k\Delta\theta \cdot \sec^2 \alpha, \qquad (\theta \lesseqgtr \alpha \lesseqgtr \theta \pm \Delta\theta),$$

where E_C is the error in the computed value of the current, and $\Delta\theta$ is the error made in reading the angle θ. Since $\Delta\theta$ is very small, $E_C = k \sec^2\theta \cdot \Delta\theta$, approximately. The error E_C is extremely large if θ is near 90°, even if $\Delta\theta$ is small; hence this form of galvanometer is not used in accurate work.

EXERCISES LIII. — INCREMENTS LAW OF THE MEAN

1. At what point on the parabola $y = x^2$ is the tangent parallel to the secant drawn through the points where $x = 1$ and $x = 2$?

2. Proceed as in Ex. 1 for the curve $y = \sin x$, and the points where $x = 30°$ and $x = 31°$.

3. Proceed as in Ex. 1 for the curve $y = \log(1 + x)$, for $x = .5$ and $x = .6$.

4. Discuss the differences in a four-place table of natural sines, the argument interval being 10′.

5. Proceed as in Ex. 4 for a similar table of natural cosines; of natural tangents.

6. Discuss the differences in a four-place table of logarithmic sines, the entries being given for intervals of 10′.

7. Proceed as in Ex. 6 for a table of logarithmic tangents.

8. Calculate the difference in a seven-place table of $\log_{10} \sin x$ at the place where $x = 30°$; where $x = 60°$; where $x = 85°$.

9. Discuss the effect of a small change in x on the function $y = \log(1 + 1/x)$.

10. If $\log_{10} N = 1.2070 \pm .0002$, what is the uncertainty in N? [The term $\pm .0002$ indicates the uncertainty in the value 1.2070.]

11. If the angle of elevation of a mountain peak, as measured from a point in the plain 5 mi. distant from it, is $5° 20' \pm 2'$, what is the uncertainty in the computed height of the peak?

12. The horizontal range of a gun is $R = (V^2/g) \sin 2\alpha$, where V is the muzzle speed and α the angle of elevation of the gun. If $V = 1200$ ft./sec., discuss the effect upon R of an error of 5′ in the angle of elevation.

13. The distance to the sea horizon from a point h ft. above sea level is $D = \sqrt{2Rh + h^2}$, where R is the radius of the earth. Discuss the change in D due to a change of one foot in h. (D, R, and h are all to be taken in the same units.) If D is tabulated for values of h at intervals of one foot, what is the tabular difference at the place where $h = 60$?

14. If the boiling point of water at height H ft. above sea level is T, $H = 517\ (212° - T) - (212° - T)^2$, T being the boiling temperature in degrees F. Discuss the uncertainty in H, if T can be measured to 1°. If H be tabulated with argument T at intervals of 1°, what is the tabular entry and the tabular difference when $T = 200°$?

15. When a pendulum of length l (feet) swings through a small angle α (radians), the time (seconds) of one swing is $T = \pi\sqrt{l/g}\,(1 + \alpha^2/16)$. What is the effect on T of a change in α, say from 5° to 6° ? Of a change in l from 36 in. to 37 in. ? Of a change in g from 32.16 to 32.2 ?

16. The viscosity of water at $\theta°$ C. is $P = 1/(1 + .0337\,\theta + .00022\,\theta^2)$. Discuss the change in P due to a small change in θ. What is the average value of P from $\theta = 20°$ to $\theta = 30°$?

17. The quantity of heat (measured in calories) required to raise one kgm. of water from 0° C. to $\theta°$ C. is $H = 94.21\,(365 - \theta)^{0.3125} + k$. How much heat is required to raise the temperature of one kgm. of water 1° C. when $\theta = 10°$? 20° ? 30° ? 70° ? To find k, observe that $H = 0$ when $\theta = 0$.

18. The coefficient of friction of water flowing through a pipe of diameter D (inches) with a speed V (ft./sec.) is $f = .0126 + (.0315 - .06\,D)/\sqrt{V}$. What is the effect on f of a small change in V? in D?

19. If the values of $\int_0^x \sqrt{1 - .2\sin^2 x}\,dx$ were tabulated with x as argument, for every degree, what would be the tabular difference at the place in the table where $x = 30°$? See *Tables*, V, E.

133. Limit of Error. In using the formula [B] the uncertainty in the value of c is troublesome. If the value of dy/dx at $x = a$ is used in place of its value at $x = c$, the error made in finding Δy by [B] can be expressed in terms of the second derivative d^2y/dx^2.

We shall use the convenient notation mentioned in Ex. 33, p. 57, and Ex. 5, p. 222, for the derivatives of $f(x)$:

$$f'(x) = \frac{df(x)}{dx} = \frac{dy}{dx} \text{ (the slope of } y = f(x)).$$

$$f''(x) = \frac{d^2f(x)}{dx^2} = \frac{d^2y}{dx^2} = \frac{df'(x)}{dx} \text{ (the flexion).}$$

Let M_2 denote the maximum of the numerical value of $f''(x)$ between two points $x = a$ and $x = b$, so that

$$(1) \qquad |f''(x)| \leqq M_2.$$

The area under the curve $y = f''(x)$ between $x = a$ and any point $x = x$ between a and b is evidently not greater than the area under the horizontal line $y = M_2$; that is, if $a < x < b$,

$$(2) \quad \left|\int_{x=a}^{x=x} f''(x)\,dx\right| \leqq \int_{x=a}^{x=x} M_2\,dx,$$

or

$$\left| f'(x)\Big]_{x=a}^{x=x} \right| \leqq M_2\, x\Big]_{x=a}^{x=x},$$

since $df'(x)/dx = f''(x)$, and M_2 is a constant; whence substituting the limits of integration in the usual manner:

$$(3) \quad |f'(x) - f'(a)| \leqq M_2(x - a),$$

which is geometrically shown in Fig. 60. It follows that the area under the curve $y = f'(x) - f'(a)$ is not greater than that under the line $y = M_2(x - a)$:

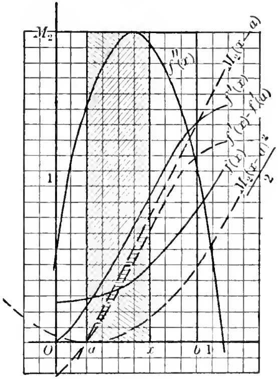


Fig. 60

$$(4) \qquad \left|\int_{x=a}^{x=x} [f'(x) - f'(a)]\,dx\right| \leqq \int_{x=a}^{x=x} M_2(x - a)\,dx;$$

or since $f'(a)$ and M_2 are constants and $df(x)/dx = f'(x)$,

$$\left|\left[f(x) - f'(a)\cdot x\right]_{x=a}^{x=x}\right| \leqq M_2 \frac{(x-a)^2}{2}\Bigg]_{x=a}^{x=x};$$

whence, substituting the limits in the usual manner,

$$[C] \qquad \left| f(x)-f(a)-f'(a)(x-a) \right| \leqq M_2\frac{(x-a)^2}{2},$$

which holds for all values of x between $x=a$ and $x=b$. This formula may be written even if $x<a$:

$$[C^*]\ f(x)=f(a)+f'(a)(x-a)+E_2, \text{ where } |E_2| \leqq M_2\frac{(x-a)^2}{2},$$

and E_2 is the error made in using $f'(a)$ in place of $f'(c)$ in formula $[C]$; for $(x-a)^2=|x-a|^2$.

It should be noticed that E_2 is exactly the error made in substituting the tangent at $x=a$ for the curve, *i.e.* it is the difference between $\Delta y\,[=f(x)-f(a)]$ and $dy\,[=f'(a)(x-a)]$ mentioned in § 31, p. 50, and shown in Fig. 12.

The formula $[B^*]$ is exactly analogous to $[C^*]$; since $\Delta y=f(x)-f(a)$ if $\Delta x=x-a$, $[B^*]$ may be written

$$[B^*] \qquad f(x)=f(a)+E_1, \qquad |E_1| \leqq M_1\cdot|x-a|.$$

Example 1. In Ex. 1, p. 252, we found for $L=\log_{10}N$,

$$\Delta L=\frac{.00043}{N} \text{ (nearly)}.$$

Applying $[C^*]$, with $f(N)=\log_{10}N$, $a=N$, $x=N+\Delta N$, $x-a=\Delta N$ $=.001$, we find

$$\Delta L=f(N+\Delta N)-f(N)=\frac{.00043}{N}+E_2, \quad |E_2|<\frac{.000001}{2}\cdot M_2,$$

where M_2 is the maximum value of $|f''(N)|=(\log_{10}e)/N^2$ between $N=1$ and $N=10$. Hence $E_2<.00000022$. The value of ΔL found before was therefore quite accurate, — absolutely accurate as far as a five-place table is concerned.

Example 2. Apply $[C^*]$ to the function $f(x)=\sin x$, with $a=0$, and show how nearly correct the values are for $x<\pi/90=2°$.

Since $f(x)=\sin x$, and $a=0$, $[C^*]$ becomes

$$\sin x=\sin(0)+\cos(0)\cdot(x-0)+E_2=x+E_2, \quad |E_2| \leqq M_2\frac{x^2}{2},$$

where M_2 is the maximum of $|f''(x)|=|-\sin x|$ between 0 and $\pi/90$, that is $M_2=\sin(\pi/90)=\sin 2°=.0349$. Hence $E_2<.0175\,x^2$. Since

$x < \pi/90$, $x^2 < \pi^2/8100 < .0013$; hence $E_2 < .000023$, and $\sin x = x$ is correct up to $x \leqq \pi/90$ within .000023.

Similarly, for $a = \pi/4$, we have, by [C^*],

$$\sin x = \frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right)\right] + E_2, \quad |E_2| \leqq M_2 \frac{(x - \pi/4)^2}{2},$$

where $M_2 < 1$. If $(x - \pi/4) < \pi/90$, $|E_2| < (\pi/90)^2 \div 2 = .0007$.

134. Extended Law of the Mean. Taylor's Theorem. The formula [C^*] can be extended very readily. Let $f'(x)$, $f''(x)$, $f'''(x)$, $\cdots f^{(n)}x$ denote the first n successive derivatives of $f(x)$:

$$f^{(k)}(x) = \frac{d^k f(x)}{dx^k} = \frac{df^{(k-1)}(x)}{dx};$$

and let the maximum of the numerical value of $f^{(n)}(x)$ from $x = a$ to $x = b$ be denoted by M_n. Then

$$|f^{(n)}(x)| \leqq M_n,$$

and
$$\left|\int_{x=a}^{x=x} f^{(n)}(x)\,dx\right| \leqq \left|\int_{x=a}^{x=x} M_n\,dx\right|, \text{ or}$$

$$|f^{(n-1)}(x) - f^{(n-1)}(a)| \leqq |M_n(x-a)|$$

for all values of x between a and b. Integrating again, we obtain, as in § 133:

$$|f^{(n-2)}(x) - f^{(n-2)}(a) - f^{(n-1)}(a)(x-a)| \leqq \left|M_n \frac{(x-a)^2}{2}\right|;$$

and, continuing this process by integrations until we reach $f(x)$, we find:

$$[\boldsymbol{D}] \quad \Big| f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \cdots$$
$$- \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}\Big| \leqq M_n \frac{|x-a|^n}{n!}$$

or,

$$[D^*] \quad f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots$$
$$+\frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}+E_n,$$

where

$$|E_n| \leqq M_n \frac{|x-a|^n}{n!},$$

and where M_n is the maximum of $|f^{(n)}(x)|$ between $x=a$ and $x=b$.

This formula is known as the **Extended Law of the Mean**, or **Taylor's Theorem**, after Taylor, who first gave such approximations as it expresses. It is one of the more important formulas of the Calculus.

In particular, if $a=0$, the formula becomes

$$[D^*]' \quad f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots$$
$$+\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}+E_n,$$

where $|E_n| \leqq M_n|x^n|/n!$ This special case of Taylor's Theorem is often called **Maclaurin's Theorem**.

The formula $[D^*]$ replaces $f(x)$ by a *polynomial* of the nth degree, with an error E_n. These polynomials are represented graphically by curves, which are usually close to the curve which represents $f(x)$ near $x=a$. See *Tables*, III, K.

Since the expression for E_n above contains $n!$ in the denominator, and since $n!$ grows astoundingly large as n grows larger, there is every prospect that E_n will become smaller for larger n; hence, usually, the polynomial curves come closer and closer to $f(x)$ as n increases, and the approximations are reasonably good farther and farther away from $x=a$. But it is never safe to trust to chance in this matter, and it is usually possible to see what *does* happen to E_n as n grows, without excessive work.

Example 1. Find an approximating polynomial of the third degree to replace $\sin x$ near $x = 0$, and determine the error in using it up to $x = \pi/18 = 10°$.

Since $f(x) = \sin x$ and $a = 0$, we have $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{\text{iv}}(x) = +\sin x$, whence $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$; and $[\text{Max.}\ |f^{\text{iv}}(x)|] = [\text{Max.}\ |\sin x|] = \sin 10° = .1736$, between $x = 0$ and $x = \pi/18 = 10°$. Hence

$$\sin x = 0 + 1 \cdot (x - 0) + 0 + (-1) \cdot \frac{(x-0)^3}{2 \cdot 3} + E_4 = x - \frac{x^3}{3\,!} + E_4,$$

where $|E_4| < (.1736) \cdot x^4/4\,! \leqq (.1736)\ (\pi/18)^4 \div 4\,! < .000007$, when x lies between 0 and $\pi/18$.

In general, the approximation grows better as n grows larger, for $|f^{(n)}(x)|$ is always either $|\sin x|$ or $|\cos x|$; hence $M_n \leqq 1$, and $|E_n| \leqq x^n/n\,!$ which diminishes very rapidly as n increases, especially if $x < 1 = 57°.3$. For $n = 7$, the formula gives, for $x > 0$,

$$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} + E_7, \quad |E_7| < x^7/7\,!.$$

EXERCISES LIV. — EXTENDED LAW OF THE MEAN

1. Apply the formula (C^*) to obtain an approximating polynomial of the first degree for $\tan x$, with $a = 0$. Show that the error, when $|x| < \pi/90$, is less than .00003. Draw a figure to show the comparison between $\tan x$ and the approximating linear function.

2. Apply $[D^*]$ to obtain an approximating quadratic for $\cos x$, with $a = 0$. Show that the error, when $|x| < \pi/10$ is less than $(\pi/10)^3 \div 3\,!$ Draw a figure.

3. Apply $[D^*]'$ to obtain an approximating cubic for $\cos x$, near $x = 0$. Hence show that the formula found in Ex. 2 is really correct, when $|x| < \pi/10$, to within $(\pi/10)^4 \div 4!$ Draw a figure.

4. Obtain an approximation of the third degree for $\sin x$ near $x = \pi/3$. Show that it is correct to within $(\pi/10)^4 \div 4\,!$ for angles which differ from $\pi/3$ by less than $\pi/10$. Draw a figure.

5. Obtain an approximation of the first degree, one of the second degree, one of the third degree, for each of the following functions near the value of x mentioned; find an upper limit of the error in each case

for values of x which differ from the valne of a by the amount specified; draw a figure showing the three approximations in each case:

(*a*) e^x, $a = 0$, $|x - a| < .1$.
(*b*) $\tan x$, $a = 0$, $|x - a| < \pi/90$.
(*c*) $\log(1+x)$, $a=0$, $|x-a|<.2$.
(*d*) $\cos x$, $a=\pi/4$, $|x-a|<\pi/18$.
(*e*) e^{-x}, $a = 2$, $|x - a| < .5$.
(*f*) $\sin x$, $a=\pi/2$, $|x-a|<\pi/45$.
(*g*) $\tan x$, $a=\pi/4$, $|x-a|<\pi/90$.
(*h*) x^2+x+1, $a=1$, $|x-a|<1/5$.
(*i*) $2x^2 - x - 1$, $a = 1/2$, $|x - a| < 1$.
(*j*) $x^3 - 2x^2 - x + 1$, $a = -2$, $|x - a| < .5$.

6. Find a polynomial which represents $\sin x$ to seven decimal places (inclusive), for $|x| < 10°$.

7. Proceed as in Ex. 6, for $\cos x$; for e^{-x}, when $0 < x < 1$.

8. Show that x differs from $\sin x$ by less than .0001 for values of x less than a certain amount; and estimate this amount as well as possible.

9. Proceed as in Ex. 8, for the expressions $1 - x^2/2$ and $\cos x$.

10. Show that the line $y = ax + b$ which passes through (0, 0) and has the same slope as $y = \sin x$ at that point, is precisely the same as the result of formula [C*].

11. Show that $1 - x^2/2$ agrees with $\cos x$ in its value, its first derivative, and its second derivative, at $x = 0$.

12. Express $\log(3/2)[= \log(1 + 1/2)]$ in powers of (1/2) so that the result shall be correct to three places.

13. What is the maximum error in the approximation $x \sin x = x^2$, when $|x| < \pi/12$?

14. Show that, near its vertex, the catenary $y = \cosh x$ has nearly the form of the parabola $y = 1 + x^2/2$. Find an upper limit of the error if $|x| < 0.1$.

15. The quantity of current C (in watts) consumed per candle power by a certain electric lamp in terms of voltage v is $C = 2.7 + 10^{8.007-.0767v}$. Express C by a polynomial in $v - 115$ correct from $v = 110$ up to $v = 120$ to within .025 watt.

135. Application of Taylor's Theorem to Extremes. If a function $y = f(x)$ is given whose maxima and minima are to be found, we may find the critical points, as in § 38, p. 63. Let a be one solution of $f'(x) = 0$, that is, a critical value. Then, since $f'(a) = 0$, we have, by [D*],

$$\Delta y = f(x) - f(a) = 0 + \frac{f''(a)}{2!}(x-a)^2 + E_3, \quad |E_3| \leqq M_3\frac{(x-a)^3}{3!},$$

where $M_3 \geqq |f'''(x)|$. Hence the sign of Δy is determined by the sign of $f''(a)$ when $(x-a)$ is sufficiently small. If $f''(a) > 0$, $\Delta y > 0$, and $f(x)$ is a **minimum** at $x = a$; if $f''(a) < 0$, $\Delta y < 0$, and $f(x)$ is a **maximum** at $x = a$. (See § 47, p. 75.)

If $f''(a) = 0$, the question is not decided.* But in that case, by $[D^*]$:

$$\Delta y = f(x) - f(a) = 0 + 0 + \frac{f'''(a)}{3\,!}(x-a)^3 + \frac{f^{\text{iv}}(a)}{4\,!}(x-a)^4 + E_5,$$

where $|E_5| \leqq M_5 |x-a|^5/5\,!$, $M_5 \geqq |f^{\text{v}}(x)|$. From this we see that if $f'''(a) \neq 0$ there is **neither** a maximum nor a minimum, for $(x-a)^3$ changes sign near $x = a$. But if $f'''(a) = 0$, then $f^{\text{iv}}(a)$ determines the sign of Δy, as in the case of $f''(a)$ above.

In general, if $f^{(k)}(a)$ is the first one of the successive derivatives, $f'(a)$, $f''(a)$, $\cdots$, which is not zero at $x = a$, then there is:

no extreme if k is odd;
a maximum if k is even and $f^{(k)}(a) < 0$;
a minimum if k is even and $f^{(k)}(a) > 0$.

Example 1. Find the extremes for $y = x^4$.

Since $f(x) = x^4$, $f'(x) = 4x^3$; hence the critical values are solutions of the equation $4x^3 = 0$, and therefore $x = 0$ is the only such critical value.

Since $f''(x) = 12x^2$, $f'''(x) = 24x$, $f^{\text{iv}}(x) = 24$, the first derivative which does not vanish at $x = 0$ is $f^{\text{iv}}(x)$, and it is positive $(=24)$. It follows that $f(x)$ is a *minimum* when $x = 0$; this is borne out by the familiar graph of the given curve.

EXERCISES LV. — EXTREMES

1. Study the extremes in the following functions:

(*a*) x^6.	(*e*) $(x+3)^5$.	(*i*) $x^2 \sin x$.
(*b*) $(x-2)^3$.	(*f*) $x^4(2x-1)^3$.	(*j*) $x^4 \cos x$.
(*c*) $4x^3 - 3x^4$.	(*g*) $\sin x^3$.	(*k*) $x^3 \tan x$.
(*d*) $x^3(1+x)^3$.	(*h*) $x - \sin x$.	(*l*) e^{-1/x^2}.

* The methods which follow are logically sound and can always be carried out when the derivatives can be found. But if several derivatives vanish (or, what is worse, fail to exist), the method of § 40, p. 64, is better in practice.

2. Discuss the extremes of the curves $y = x^n$, for all positive integral values of n.

3. Solve the problem of Ex. 18, List XIV, p. 69, by the method of § 135.

4. If a set of observed values of a quantity y which depends upon another quantity x are y_0, y_1, y_2, $\cdots$, y_n, when x has the values x_0, x_1, x_2, $\cdots$, x_n, and if y is connected with x by means of an equation of the form $y = kx$, the sum of the squares of the differences between the observed and the computed values of y is :

$$S = (y_0 - kx_0)^2 + (y_1 - kx_1)^2 + (y_2 - kx_2)^2 + \cdots + (y_n - kx_n)^2.$$

Show that the sum S, as a function of k, is least when

$$2\,x_0(y_0 - kx_0) + 2\,x_1\,(y_1 - kx_1) + \cdots + 2\,x_n\,(y_n - kx_n) = 0,$$

or
$$k = \sum_{i=0}^{i=n} x_i y_i \div \sum_{i=0}^{i=n} x_i^2.$$

[NOTE. Under the assumption of Ex. 18, p. 69, this value of k is the best compromise, or the *most probable value.*]

5. Using the result of Ex. 4, recompute the value of each of the constants of proportionality k in Exs. 18–23, p. 69.

6. An open tank is to be constructed with square base and vertical sides so as to contain 10 cu. ft. of water. Find the dimensions so that the least possible quantity of material will be needed.

7. Show that the greatest rectangle that can be inscribed in a given circle is a square.

[See Ex. 25, p. 70. Other examples from List XIV may be resolved by the process of § 135.]

8. What is the maximum contents of a cone that can be folded from a filter paper of 8 in. diameter?

9. A gutter whose cross section is an arc of a circle is to be made by bending into shape a strip of copper. If the width of the strip is a, show that the radius of the cross section when the carrying capacity is a maximum is a/π. [OSGOOD.]

10. A battery of internal resistance r and E. M. F. e sends a current through an external resistance R. The power given to the external circuit is

$$W = \frac{Re^2}{(R + r)^2}.$$

If $e = 3.3$ and $r = 1.5$, with what value of R will the greatest power be given to the external circuit? [SAXELBY.]

11. Find the shortest distance from the origin to the curve $y = a^x$; show that it is measured along a straight line from the origin to the intersection of the given curve with the curve $x = -y^2 \log a$.

12. Show that the maximum and the minimum distances from a point (a, b) to the curve $y = x^2$ join (a, b) to the intersections of $y = x^2$ with $x(y - b + 1) = a$.

136. Indeterminate Forms. The quotient of two functions is not defined at a point where the divisor is zero. Such quotients $f(x) \div \phi(x)$ at $x = a$, where $f(a) = \phi(a) = 0$, are called **indeterminate forms.*** We may note that the graph of

$$(1) \qquad q = \frac{f(x)}{\phi(x)}, \qquad (f(a) = \phi(a) = 0),$$

may be quite regular near $x = a$; hence it is natural to make the definition:

$$(2) \qquad q\Big]_{x=a} = \frac{f(x)}{\phi(x)}\Big]_{x=a} = \lim_{x \doteq a} \frac{f(x)}{\phi(x)}.$$

If we apply [D*], we obtain,

$$q = \frac{f(x)}{\phi(x)} = \frac{0 + f'(a)(x-a) + E_2'}{0 + \phi'(a)(x-a) + E_2''},$$

where

$$|E_2'| \leqq M_2'(x-a)^2/2!, \quad |E_2''| \leqq M_2''(x-a)^2/2!,$$

and $\quad M_2' \geqq |f''(x)|, \quad M_2'' \geqq |\phi''(x)|$, near $x = a$.

Hence $$q = \frac{f(x)}{\phi(x)} = \frac{f'(a) + p'M_2'\dfrac{(x-a)}{2}}{\phi'(a) + p''M_2''\dfrac{(x-a)}{2}}$$

* If $\phi(a) = 0$ but $f(a) \neq 0$ the quotient q evidently becomes infinite; in that case the graph of (1) shows a vertical asymptote.

where p' and p'' are numbers between -1 and $+1$. It follows that

$$(3) \qquad \lim_{x \doteq a} q = \lim_{x \doteq a} \frac{f(x)}{\phi(x)} = \frac{f'(a)}{\phi'(a)}$$

unless $\phi'(a) = 0$. But if $\phi'(a) = 0$, q becomes infinite, and the graph of (1) has a *vertical asymptote* at $x = a$ unless $f'(a) = 0$ also. If both $f'(a)$ and $\phi'(a)$ are zero, it follows in precisely the same manner as above, that

$$q = \frac{f(x)}{\phi(x)} = \frac{f^{(k)}(a) + p' M'_{k+1} \dfrac{(x-a)}{(k+1)!}}{\phi^{(k)}(a) + p'' M'_{k+1} \dfrac{(x-a)}{(k+1)!}},$$

where either $f^{(k)}(a)$ or $\phi^{(k)}(a)$ is not zero, but all preceding derivatives of both $f(x)$ and $\phi(x)$ are zero at $x = a$; and where $M'_{k+1} \geqq |f^{(k+1)}(x)|$, $M''_{k+1} \geqq |\phi^{(k+1)}(x)|$ near $x = a$ and where p' and p'' are numbers between -1 and $+1$. It follows that

$$\lim_{x \doteq a} q = \lim_{x \doteq a} \frac{f(x)}{\phi(x)} \equiv \frac{f^{(k)}(a)}{\phi^{(k)}(a)},$$

provided all previous derivatives of both $f(x)$ and $\phi(x)$ are zero at $x = a$, and provided $\phi^{(k)}(a) \neq 0$. If $\phi^{(k)}(a) = 0$, $f^{(k)}(a) \neq 0$, then q becomes infinite and the graph of (1) has a *vertical asymptote* at $x = a$.

It should be noted that (3) is only a repetition of Rule [VII], p. 36. For if $u = f(x)$ and $v = \phi(x)$, since $f(a) = \phi(a) = 0$,

$$q = \frac{f(x)}{\phi(x)} = \frac{f(x) - f(a)}{\phi(x) - \phi(a)} = \frac{\Delta u}{\Delta v} = \frac{\Delta u}{\Delta x} \div \frac{\Delta v}{\Delta x},$$

where $\Delta x = x - a$; and therefore

$$\lim_{x \doteq a} q = \lim_{\Delta x \doteq 0} \frac{\Delta u}{\Delta x} \div \lim_{\Delta x \doteq 0} \frac{\Delta v}{\Delta x} = \left[\frac{du}{dx} \div \frac{dv}{dx}\right]_{x=a} = \left[\frac{f'(x)}{\phi'(x)}\right]_{x=a} = \frac{f'(a)}{\phi'(a)},$$

provided $\phi'(a)$ is not zero (see Theorem D, p. 18).

Example 1. To find $\lim\limits_{x \doteq 0} [(\tan x) \div x]$.

Here $f(x) = \tan x$, $\phi(x) = x$; $f(0) = \phi(0) = 0$; hence

$$\lim_{x \doteq 0} \frac{\tan x}{x} = \frac{f'(0)}{\phi'(0)} = \frac{[\sec^2 x]_{x \doteq 0}}{1} = 1.$$

Draw the graph $q = (\tan x) \div x$ and notice that this value $q = 1$ fits exactly where $x = 0$.

This limit can be found directly as follows:

$$\lim_{h \doteq 0} \frac{\tan h}{h} = \lim_{h \doteq 0} \frac{\tan(0 + h) - \tan(0)}{(0 + h) - (0)} = \frac{d \tan x}{dx}\bigg]_{x=0} = \sec^2 x\bigg]_{x=0} = 1.$$

Compare the work done in § 96, p. 167, for $\lim (\cos \Delta\theta - 1)/\Delta\theta$.

Example 2. To find $\lim\limits_{x \doteq 0} (1 - \cos x)/x^2$.

Here $f(x) = 1 - \cos x$, $\phi(x) = x^2$; $f(0) = \phi(0) = 0$; $f'(0) = \sin(0) = 0$ and $\phi'(0) = 0$; $f''(x) = \cos x$, $\phi''(x) = 2$; hence

$$\lim_{x \doteq 0} \frac{1 - \cos x}{x^2} = \frac{\cos x}{2}\bigg]_{x=0} = \frac{1}{2}.$$

Draw the graph of $q = (1 - \cos x)/x^2$, and note that $(x = 0,\ q = 1/2)$ fits it well.

137. Infinitesimals of Higher Order. When the quotient

$$q = \frac{f(x)}{x^n} \tag{1}$$

approaches a finite number not zero when x is infinitesimal:

$$\lim_{x \doteq 0} q = \lim_{x \doteq 0} \frac{f(x)}{x^n} = k \neq 0, \tag{2}$$

then $f(x)$ is said **to be an infinitesimal of order n with respect to x.** An infinitesimal whose order is greater than 1 is called an **infinitesimal of higher order.**

The equation (2) may be reduced to the form

$$\lim_{x \doteq 0} [f(x) - kx^n] = 0, \tag{3}$$

or

$$f(x) = (k + E)\, x^n, \tag{4}$$

where $\lim E = 0$. The quantity kx^n is called the **principal part** of the infinitesimal $f(x)$. The difference $f(x) - kx^n = Ex^n$ is evidently an infinitesimal whose order is greater than n, for

$$\lim (Ex^n \div x^n) = \lim E = 0.$$

Thus by Example 2, p. 265, $1 - \cos x$ is an infinitesimal of the 2d order with respect to x; its principal part is $x^2/2$. Note that

$$1 - \cos x = x^2/2 + px^3/3\,!,$$

by [D*], where $-1 \leqq p \leqq +1$; the principal part is the first term of Taylor's Theorem that does not vanish.

In general, if $f(0) = f'(0) = f''(0) = \cdots = f^{(k-1)}(0) = 0$, but $f^{(k)}(0) \neq 0$, the formula [D*] gives, for $a = 0$,

$$f(x) = f^{(k)}(0) \cdot x^k/k\,! + pM_{k+1}x^{k+1}/(k+1)\,!$$

where $M_{k+1} \geqq |f^{(k+1)}(x)|$ near $x = 0$, and $-1 \leqq p \leqq +1$. Hence $f(x)$ is an infinitesimal of order k with respect to x, and its principal part is $f^{(k)}(0)x^k/k\,!$.

EXERCISES LVI. — INDETERMINATE FORMS INFINITESIMALS

1. Evaluate the indeterminate forms below, in which the notation $\phi(x)\,|_a$ means to determine the limit of $\phi(x)$ when $x \doteq a$. The vertical bar applies to all that precedes it. Draw the graphs as in Exs. 1, 2, above.

(a) $\sin x/x\,|_0$. (b) $(\tan 2x)/x\,|_0$. (c) $\sin ax/\sin bx\,|_0$.

(d) $\left.\dfrac{x^n - 1}{x - 1}\right|_1$. (e) $\left.\dfrac{e^x - 1}{x}\right|_0$. (f) $\left.\dfrac{a^x - 1}{x}\right|_0$. (g) $\left.\dfrac{\tan x}{\tan 3x}\right|_0$.

(h) $\left.\dfrac{a^x - b^x}{x}\right|_0$. (k) $\left.\dfrac{\sin x}{\log(1+x)}\right|_0$. (n) $\left.\dfrac{e^x - e^{-x}}{\sqrt{x}}\right|_0$.

(i) $\left.\dfrac{\log x}{\sqrt{x^2 - 1}}\right|_1$. (l) $\left.\dfrac{x - \sin x}{x - \tan x}\right|_0$. (o) $\left.\dfrac{x}{\log x}\right|_0$.

(j) $\left.\dfrac{x \cos x - \sin x}{x}\right|_0$. (m) $\left.\dfrac{\sin^{-1} x}{\tan^{-1} x}\right|_0$. (p) $\left.\dfrac{e^x - e^{\sin x}}{x - \sin x}\right|_0$.

(q) $\left.\dfrac{a^{\log x} - x}{\log x}\right|_1$. (t) $\left.\dfrac{\log(x^2 - 3)}{x^2 + 3x - 10}\right|_2$.

(r) $\left.\dfrac{\sin x}{\sin 2x}\right|_\pi$. (u) $\left.\dfrac{\sin^{-1}(2 - x)}{\sqrt{x^2 - 3x + 2}}\right|_2$.

(s) $\left.\dfrac{2 \sin x - 1}{\sin 6x}\right|_{\pi/6}$. (v) $\left.\dfrac{\sin^{-1}(\sqrt{a^2 - x^2}/a)}{\sqrt{a^2 - x^2}}\right|_a$.

2. Determine the order of each of the quantities below when the variable x is the standard infinitesimal:

(*a*) $x - \sin x$.	(*e*) $e^x - e^{\sin x}$.	(*i*) $\sin 2x - 2\sin x$.
(*b*) $e^x - e^{-x}$.	(*f*) $a^x - 1$.	(*j*) $\log\cos x$.
(*c*) $x^2 \sin x^2$.	(*g*) $\log[(a+x)/(a-x)]$.	(*k*) $\log(1 + e^{-1/x})$.
(*d*) $\log(1+x) - x$.	(*h*) $x\cos x - \sin x$.	(*l*) $\tan^{-1} x - \sin^{-1} x$.
(*m*) $\log\cos x - \sin^2 x$.	(*n*) $2x - e^x + e^{-x}$.	(*o*) $\cos^{-1}(1-x) - \sqrt{2x - x^2}$.

3. Show that Ex. 1 (*a*) can be expressed as the derivative of $\sin x$ at $x = 0$, as in Example 1, p. 265.

4. Show that Exs. 1 (*e*), (*f*), (*h*), (*j*) can be expressed as the derivatives of the numerators, for $x = 0$.

5. Show that Ex. 1 (*d*) can be expressed as the derivative of its numerator divided by the derivative of its denominator, at $x = 1$.

6. Find the limit of the ratio of the surface of a sphere to its volume, as the radius approaches zero.

7. Find the limit of the ratio of a chord of a circle to the distance along a radius perpendicular to the chord from the chord to the circle.

8. Given two quantities u and v which vary with the time t, so that $u = f(t)$ and $v = \phi(t)$, show that

$$\lim_{\Delta v \doteq 0} \frac{\Delta u}{\Delta v} = \left[\lim_{\Delta t \doteq 0} \frac{\Delta u}{\Delta t}\right] \div \left[\lim_{\Delta t \doteq 0} \frac{\Delta v}{\Delta t}\right].$$

9. Show that the slope of the path of a moving body is the ratio of its vertical speed to its horizontal speed.

138. Double Law of Mean. Let $y = f(x)$ and $y = \phi(x)$ be two simple smooth curves between $x = a$ and $x = b$; and let us draw the two secants which cut these two curves at the points $x = a$ and $x = b$. Then there exists a point c such that the ratio of the slopes of the curves at $x = c$ is equal to the ratio of the slopes of the secants.

$$(1) \qquad \frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(c)}{\phi'(c)}, \quad a < c < b.$$

To prove this, consider the parallel curves

$$y = f(x) - f(a) \quad \text{and} \quad y = \phi(x) - \phi(a),$$

which both go through the same left-hand point $(a, 0)$ and the ratio of the slopes of whose secants is, as above, $[f(b) - f(a)]/[\phi(b) - \phi(a)]$.

Multiplying all the ordinates of $y = \phi(x) - \phi(a)$ by this ratio, we have the new curve

$$y = \frac{f(b) - f(a)}{\phi(b) - \phi(a)}[\phi(x) - \phi(a)] = F(x), \text{ (say).}$$

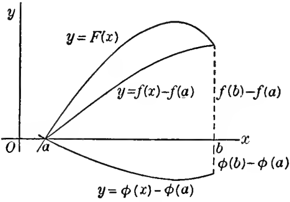


Fig. 61

This curve has both the same left-hand point $(a, 0)$ and the same right-hand point

$$[b, f(b) - f(a)]$$

as the curve

$$y = f(x) - f(a)$$

Hence, by Rolle's Theorem, there is a point $x = c$, for which the two have the same slope; that is, such that

$$f'(c) = F'(c),$$

since $f(x) - F(x)$ vanishes at $x = a$ and $x = b$. Replacing F by its value above, the statement (1) is proved.

139. The Indeterminate Form $\infty \div \infty$. Vertical Asymptotes. Suppose the curves $y = f(x)$ and $y = \phi(x)$ are continuous and have a continuous slope from $x = a$ up to but not including $x = A$, where both ordinates become infinite; that is, *each curve has a vertical asymptote at* $x = A$. Then b and a can be so chosen in the neighborhood of A that both $f(a)/f(b)$ and $\phi(a)/\phi(b)$ shall be as small as we please. For however close to A one takes a, $f(a)$ and $\phi(a)$ are finite. Taking now b between a and A, we can give $f(b)$ any value above $f(a)$. Therefore the first of the preceding ratios (and in like manner the second) can by proper choice of b, after any choice of a, be made as small as one pleases. Notice that a and b must be on the same side of the vertical asymptote.

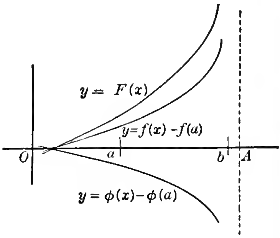


Fig. 62

Let the choice of a and b be made as just indicated. The theorem of § 138 still holds:

$$\frac{f'(c)}{\phi'(c)} = \frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{\left[1-\dfrac{f(a)}{f(b)}\right]f(b)}{\left[1-\dfrac{\phi(a)}{\phi(b)}\right]\phi(b)}, \quad (a<c<b),$$

which, by the preceding remark, approaches $f(b)/\phi(b)$ as b approaches A.

If we let a approach A, the preceding conditions insure that c and b will both approach A. Thus, finally,

$$\lim_{x \doteq A} \frac{f'(x)}{\phi'(x)} = \lim_{x \doteq A} \frac{f(x)}{\phi(x)};$$

that is, if $f(x)$ and $\phi(x)$ each becomes infinite at $x = A$, and if the quotient $f'(x)/\phi'(x)$ approaches a limit as $x \doteq A$ from either side, then $f(x)/\phi(x)$ approaches the same limit.

Similarly, if $f'(x)/\phi'(x)$ assumes the form $\infty \div \infty$, and if $f''(x)/\phi''(x)$ approaches a limit, then $f'(x)/\phi'(x)$, and hence also $f(x)/\phi(x)$ approaches the same limit, and so on.

Since $f(x)/\phi(x) = [1/\phi(x)] \div [1/f(x)]$, any fraction that takes one of the two forms $0/0$, $\infty \div \infty$, can also be put into the other form. In practice this method may be more convenient or less so, than the preceding one, depending upon the particular example. Thus, as $x \doteq \pi/2$, $\tan x$ and $\sec x$ both become infinite, while $\operatorname{ctn} x$ and $\cos x$ approach zero; hence

$$\lim_{x \doteq \pi/2} \frac{\tan x}{\sec x} = \lim_{x \doteq \pi/2} \frac{\cos x}{\operatorname{ctn} x} = 1.$$

140. Other Indeterminate Forms. Likewise, if $f(x) \doteq 0$ as $\phi(x)$ becomes infinite, their product is of the form $0 \times \infty$, and it can be put into either of the preceding forms.

Thus, as $x \doteq 0$, $\log x$ becomes $-\infty$; so that

$$\lim_{x \doteq 0} (x \log x) = \lim_{x \doteq 0} \frac{\log x}{1/x} = \lim_{x \doteq 0} \frac{1/x}{-1/x^2} = \lim_{x \doteq 0} (-x) = 0.$$

Other indeterminate forms are $\infty - \infty$, 1^∞, 0^0, ∞^0. All these can be made to depend on the forms already considered. For let α, β, γ, δ, ϵ, be variables simultaneously approaching, respectively, ∞, ∞, 1, 0, 0. Then $\alpha - \beta$, γ^α, δ^ϵ, α^ϵ take, respectively, the preceding four indeterminate forms. But

$$\lim (\alpha - \beta) = \lim \frac{1/\beta - 1/\alpha}{1/\beta\alpha},$$

which is of the form $0/0$; while the logarithms of the others,

$$\log \gamma^{\alpha} = \alpha \log \gamma, \quad \log \delta^{\epsilon} = \epsilon \log \delta, \quad \log \alpha^{\epsilon} = \epsilon \log \alpha,$$

are each of the form $0 \times \infty$.

Example 1. Thus, when $x \doteq \pi/2$, $(\sin x)^{\tan x}$ takes the form 1^{∞}. But

$$(\sin x)^{\tan x} = e^{[\log \sin x]/\operatorname{ctn} x},$$

which approaches the same limit as $e^{-\operatorname{ctn} x/\csc^2 x}$, as $x \doteq \pi/2$, and this limit is evidently $e^0 = 1$.

Example 2. Similarly, when x becomes infinite, $(1/x)^{1/(2x+1)}$ takes the form 0^0. It may be written in the form,

$$e^{[-\log x]/(2x+1)},$$

which approaches the same limit as $e^{-1/2x}$, that is, the limit is $e^0 = 1$, as $x \doteq \infty$.

Example 3. As an example of the last form, ∞^0, take $(1/x)^x$ as $x \doteq 0$. This becomes

$$e^{-x \log x},$$

and approaches $e^0 = 1$, as $x \doteq 0$.

Indeterminate forms in two variables cannot be evaluated, unless one knows a law connecting the variables as they approach their limits, which practically reduces the problem to a problem in one letter.

EXERCISES LVII. — SECONDARY INDETERMINATE FORMS

1. Evaluate each of the following indeterminate forms, where $\phi(x)|_a$ means the limit of $\phi(x)$ as x approaches a. Draw a graph in each case.

(*a*) $\left.\dfrac{x}{e^x}\right|_{\infty}$. (*g*) $\left.\dfrac{x^n}{e^x}\right|_{\infty}$. (*m*) $x^{\sin x}\,|_0$.

(*b*) $\left.\dfrac{\log \operatorname{ctn} x}{\log \cos x}\right|_{\pi/2}$. (*h*) $\left.\dfrac{\log x}{e^x}\right|_{\infty}$. (*n*) $(1+x)^{1/x}\,|_0$.

(*c*) $\left.\dfrac{\tan x}{\log(\pi/2 - x)}\right|_{\pi/2}$. (*i*) $\left.\dfrac{\log z}{z}\right|_{\infty}$. (*o*) $(1+1/x)^x\,|_{\infty}$.

(*d*) $\left.\dfrac{x^2}{e^x}\right|_{\infty}$. (*j*) $x \operatorname{ctn} x\,|_0$. (*p*) $(\tan x)^{\cos x}\,|_{\pi/2}$.

(*e*) $\left.\dfrac{\log \cos x}{\sin^2 x}\right|_0$. (*k*) $x^2 \log x^3\,|_0$. (*q*) $(\sin x)^{\sin x}\,|_0$.

(*f*) $\left.\dfrac{\log \sin x}{\log \tan x}\right|_0$. (*l*) $(\tan x - \sec x)\,|_{\pi/2}$. (*r*) $\left.\left(\tan x - \dfrac{1}{\pi/2 - x}\right)\right|_{\pi/2}$.

2. If $\phi(x) = 2 - 2\cosh x + x \sinh x$, show that

$$[\phi'(x)/\phi(x)]\,|_0 = \infty \text{ and } [\phi'(x)/\phi(x)]\,|_\infty = 1.$$

3. Find the limit, as x becomes infinite, of the product $x^n e^{-kx}$ for any positive integral value of n. Draw the graphs for the cases $n = 1, 2, 3$. Hence show that the damping factor e^{-x} reduces the curve $y = x^n$ to a new curve asymptotic to the x-axis.

4 Show that the improper integral of xe^{-x} from 0 to ∞ exists, and that its value is 1. [See *Tables*, IV, 97 *a*, 109 ; and V, *F*.]

5. Show that x^m, where m is fractional, lies between two integral powers of x. Hence show that the curve $y = x^m e^{-x}$ is asymptotic to the x-axis.

6. Show that the improper integral of $x^m e^{-x}$ from $x = 0$ to $x = \infty$, where m is any positive fraction, exists, by use of Ex. 5.

7. Find the value of each of the following improper integrals from Table V, *F*:

(1) $\displaystyle\int_0^\infty x^2 e^{-x}\,dx.$ (3) $\displaystyle\int_0^\infty xe^{-2x}\,dx.$ (5) $\displaystyle\int_0^\infty x^{1.2} e^{-3x}\,dx.$

(2) $\displaystyle\int_0^\infty x^{0.2} e^{-x}\,dx.$ (4) $\displaystyle\int_0^\infty x^{2.3} e^{-x}\,dx.$ (6) $\displaystyle\int_0^\infty x^{4.2} e^{-2x}\,dx.$

8. Show that the derivative of $\log x$ is

$$\frac{d \log x}{dx} = \lim_{h \doteq 0} \frac{\log (x + h) - \log x}{h} = \lim_{h \doteq 0} \left[\frac{1}{h} \log\left(1 + \frac{h}{x}\right)\right]$$
$$= \frac{1}{x} \lim_{z \doteq 0} \log (1 + z)^{1/z}, \text{ where } z = \frac{h}{x}.$$

Hence show, by use of Ex. 1 (n), that the derivative in question is $1/x$.

9. Throw the expression of Ex. 8 into the form

$$\frac{1}{x} \lim_{h \doteq 0} \left[\frac{x}{h} \log\left(1 + \frac{h}{x}\right)\right] = \frac{1}{x} \lim_{u \doteq 1} \frac{\log u}{u - 1},$$

where $u = 1 + h/x$. Show that the last limit above is equal to 1 ; hence verify the result of Ex. 8.

141. Infinite Series. An **infinite series** is an indicated sum of an unending sequence of terms:

(1) $$a_0 + a_1 + a_2 + \cdots + a_n + \cdots;$$

this has no meaning whatever until we make a definition, for it is impossible to *add* all these terms. Let us take the sum of the first n terms:

$$s_n = a_0 + a_1 + a_2 + \cdots + a_{n-1},$$

which is perfectly finite; if the limit of s_n exists as n becomes infinite, that limit is called the **sum of the series** (1):

$$(2) \qquad S = \lim_{n \doteq \infty} s_n = a_0 + a_1 + \cdots + a_n + \cdots$$

If $\lim\limits_{n \doteq \infty} s_n = S$ exists, the series is called **convergent**; if S does not exist, the series is called **divergent**; if the series formed by taking the numerical (or absolute) values of the terms of (1) converges, then (1) is called **absolutely convergent.** Infinite series which converge absolutely are most convenient in actual practice, for extreme precaution is necessary in dealing with other series. (See § 143, p. 276.)

Example 1. The series $1 + r + r^2 + \cdots + r^n + \cdots$ is called a **geometric series**; the number r is called the **common ratio.** A geometric series *converges absolutely for any value of r numerically less than* 1; for

$$s_n = 1 + r + r^2 + \cdots + r^{n-1} = \frac{1}{1-r} - \frac{r^n}{1-r},$$

hence $$\lim_{n \doteq \infty}\left|\frac{1}{1-r} - s_n\right| = \lim_{n \doteq \infty}\left|\frac{r^n}{1-r}\right| = 0, \text{ if } |r| < 1,$$

since r^n decreases below any number we might name as n becomes infinite. It follows that the **sum** S of the infinite series is

$$S = \lim s_n = \frac{1}{1-r}, \text{ if } |r| < 1;$$

and it is easy to see that the series still converges if r is negative, when r is replaced by its numerical value $|r|$.

Example 2. Any series $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$ of positive numbers can be compared with the geometric series of Ex. 1. Let

$$\sigma_n = a_0 + a_1 + a_2 + \cdots + a_{n-1};$$

then it is evident that σ_n increases with n. Comparing with the geometric series $a_0(1 + r + r^2 + \cdots + r^n + \cdots)$, it is clear that if

$$a_n \leqq a_0 r^n, \quad \sigma_n \leqq a_0 s_n,$$

where $s_n = 1 + r + \cdots + r^{n-1}$. *Hence* σ_n *approaches a limit if* s_n *does*, i.e. *if* $0 < r < 1$. It follows that the given series converges if a value of $r < 1$ can be found for which $a_n \leqq a_0 r^n$, that is, for which $a_n \div a_{n-1} \leqq r < 1$. There are, however, some convergent series for which this test cannot be applied satisfactorily. It may be applied in testing any series for absolute convergence; or in testing any series of positive terms. For example, consider the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots;$$

here $a_n = 1/n!$, $a_{n-1} = 1/(n-1)!$, and therefore $a_n/a_{n-1} = (n-1)!/n! = 1/n$. Hence $a_n/a_{n-1} < 1/2$ when $n \geqq 2$,

$$\sigma_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} > 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-2}}\right)$$
$$= 1 + s_{n-1},$$

where $s_{n-1} = 1 + r + \cdots + r^{n-2}$, $r = 1/2$. It follows that the given series converges and that its sum is less than $1 + 2 = 3$. [Compare § 143, p. 278; it results that $e < 3$. Compare Ex. 2, p. 275.]

142. Taylor Series. General Convergence Test. Series which resemble the geometric series except for the insertion of constant coefficients of the powers of r,

(1) $$A + Br + Cr^2 + Dr^3 + \cdots,$$

arise through application of Taylor's Theorem $[D^*]$(§ 134, p. 258); such series are called **Taylor series** or **power series.** The properties of a Taylor series are, like those of a geometric series, comparatively simple. Comparing (1) with $[D^*]$, we see that r takes the place of $(x-a)$, while $A, B, C, D, \cdots$ have the values:

$$A = f(a),\ B = \frac{f'(a)}{1!},\ C = \frac{f''(a)}{2!},\ D = \frac{f'''(a)}{3!} \cdots.$$

If we consider the sum of n such terms:

$$s_n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1},$$

we see by [D^*], that

$$f(x) = s_n + E_n, \text{ where } |E_n| \leqq M_n \frac{|x-a|^n}{n!},\ M_n \geqq |f^{(n)}(x)|;$$

or

$$s_n = f(x) - E_n.$$

It follows that **if E_n approaches zero as n becomes infinite, the infinite Taylor Series**

$$[D^{**}] \quad f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots$$

converges, and its sum is $S = \lim s_n = f(x)$.*

This is certainly true, for example, whenever $|f^{(n)}(x)|$ remains, for all values of n, less than some constant C, however large, for all values of x between $x = a$ and $x = b$. For in that case

$$\lim_{n \doteq \infty} |E_n| \leqq \lim_{n \doteq \infty} C \cdot \frac{|x-a|^n}{n!} = C \lim_{n \doteq \infty} \frac{|x-a|^n}{n!} = 0,$$

for all values of $(x-a)$.† When $|f^{(n)}(x)|$ grows larger and larger without a bound as n becomes infinite, we may still often make $|E_n|$ approach zero by making $(x-a)$ numerically small.

Example 1. Derive an infinite Taylor series in powers of x for the function $f(x) = \sin x$.

Since $f(x) = \sin x$, we have $f'(x) = \cos x$, $f''(x) = -\sin x$, and, in general, $f^{(n)}(x) = \pm \sin x$, or $\pm \cos x$; hence

$$|f^n(x)| \leqq 1, \quad \lim_{n \doteq \infty} |E_n| \leqq \lim_{n \doteq \infty} \frac{x^n}{n!} = 0;$$

* This result is forecasted in § 134, p. 258.

† This results from the fact that n eventually exceeds $(x-a)$ numerically; afterwards an increase in n diminishes the value of E_n more and more rapidly as n grows.

therefore the infinite series [D^{**}] for $a=0$ is

$$\sin x = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + \cdots;$$

this series certainly converges and its sum is $\sin x$ for all values of x, since $\lim |E_n| = 0$.

Example 2. Derive an infinite series for e^x in powers of $(x-2)$.

Since $f(x)=e^x$, we have $f'(x)=e^x, \cdots, f^{(n)}(x)=e^x$; hence $f(2)=e^2$, $f'(2)=e^2, \cdots, f^{(n)}(2)=e^2$, and $|f^{(n)}(x)| \leqq e^b$ where b is the largest value of x we shall consider. Then the series

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \cdots + \frac{e^2}{n!}(x-2)^n + \cdots$$

$$= e^2\left[1 + (x-2) + \frac{1}{2!}(x-2)^2 + \cdots + \frac{1}{n!}(x-2)^n + \cdots\right]$$

converges and its sum is e^x, for all values of x less than b; for

$$\lim_{n \doteq \infty} |E_n| \leqq \lim_{n \doteq \infty} \frac{e^b |x-2|^n}{n!} = 0.$$

Since b is any number we please, the series is convergent and its sum is e^x for all values of x.

EXERCISES LVIII. — TAYLOR SERIES

1. Obtain the infinite Taylor series for $\cos x$ in powers of x. Show that $\lim |E_n| = 0$.

2. Derive the following series, and account, when possible, for the fact that $\lim |E_n| = 0$:

(*a*) $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$; (all x).
(*b*) $e^{-x} = 1 - x + x^2/2! - x^3/3! + \cdots$; (all x).
(*c*) $\tan x = x + x^3/3 + 2x^5/15 + 17x^7/315 + \cdots$; $(|x| < \pi/2)$.
(*d*) $\log(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$; $(|x| < 1)$.
(*e*) $\sinh x = (e^x - e^{-x})/2 = x + x^3/3! + x^5/5! + \cdots$; (all x).
(*f*) $\cosh x = (e^x + e^{-x})/2 = 1 + x^2/2! + x^4/4! + \cdots$; (all x).
(*g*) $\tanh x = \sinh x/\cosh x = x - x^3/3 + 2x^5/15 - 17x^7/315 + \cdots$; (all x).

3. Show that the series of Ex. 2 (*e*) can be obtained from those of Exs. 2 (*a*) and 2 (*b*) if the terms are combined separately.

4. Show that the series of Ex. 2 (*b*) results from the series of Ex. 2 (*a*) if x is replaced by $-x$.

5. Obtain the series for $\sin x$ in powers of $(x - \pi/4)$.

6. Obtain the series for e^x in terms of powers of $(x - 1)$.

7. Obtain the series for $\log x$ in powers of $(x - 1)$. Compare it with the series of Ex. 2 (d).

8. Obtain the series for $\log (1 - x)$ in powers of x, directly; also by replacing x by $-x$ in Ex. 2(d).

9. Using the fact that $\log [(1+x)/(1-x)] = \log (1+x) - \log (1-x)$, obtain the series for $\log [(1 + x)/(1 - x)]$ by combining the separate terms of the two series of Ex. 8 and of Ex. 2 (d). This series is actually used for computing logarithms.

10. Show that the terms of the expansion of $(a + x)^n$ in powers of x are precisely those of the usual binomial theorem.

11. Show that the series for e^{a+x} in powers of x is the same as the series for e^x all multiplied by e^a.

12. Show that the series for 10^x is the same as the series for e^x with x replaced by x/M, where $M = 2.30 \cdots$.

143. Precautions about Infinite Series. There are several popular misconceptions concerning infinite series which yield to very commonplace arguments.

(a) *Infinite series are never used in computation.* Contrary to a popular belief, infinite series are never used in computation, and can never be used. This is because no one can possibly *add* **all** the terms of an infinite series. What is actually done is to use a few terms (that is, a polynomial) for actual computation; one may or may not consider how much error is made in doing this, with an obvious effect on the trustworthiness of the result.

Thus we may write

$$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \cdots \pm \frac{x^{2k+1}}{(2\,k + 1)\,!} \mp \cdots \text{ (forever)};$$

but in practical computation, we decide to use a few terms, say $\sin x = x - x^3/3\,! + x^5/5\,!$. The error in doing this can be estimated by § 134, p. 257. It is $| E_7 | \leq |x^7/7\,!|$. For reasonably small values of x [say $| x | < 14^\circ < 1/4$ (radians)], $| E_7 |$ is exceedingly small.

Many of the more useful series are so rapid in their convergence that it is really quite safe to use them without estimating the error made; but if one proceeds without any idea of how much the error amounts to, one usu-

ally computes more terms than necessary. Thus if it were required to calculate sin 14° to eight decimal places, most persons would suppose it necessary to use quite a few terms of the preceding series, if they had not estimated E_7.

(*b*) *No faith can be placed in the fact that the terms are becoming smaller.* The instinctive feeling that if the terms become quite small, one can reasonably stop and suppose the error small, is unfortunately not justified.*

Thus the series
$$\frac{1}{10}+\frac{1}{20}+\frac{1}{30}+\frac{1}{40}+\cdots+\frac{1}{10\,n}+\cdots$$
has terms which become small rather rapidly; one instinctively feels that if about one hundred terms were computed, the rest would not affect the result very much, because the next term is .001 and the succeeding ones are still smaller. *This expectation is violently wrong.*

As a matter of fact this series *diverges;* we can pass any conceivable amount by continuing the term-adding process. For

$$\tfrac{1}{30}+\tfrac{1}{40}>2\cdot\tfrac{1}{40}\ =\tfrac{1}{20},$$
$$\tfrac{1}{50}+\tfrac{1}{60}+\cdots+\tfrac{1}{80}>4\cdot\tfrac{1}{80}\ =\tfrac{1}{20},$$
$$\tfrac{1}{90}+\tfrac{1}{100}+\cdots+\tfrac{1}{160}>8\cdot\tfrac{1}{160}=\tfrac{1}{20},$$

and so on; groups of terms which total more than 1/20 continue to appear forever; twenty such groups would total over 1; 200 such groups would total over 10; and so on. The preceding series is therefore very deceptive; practically it is useless for computation, though it might appear quite promising to one who still trusted the instinctive feeling mentioned above.

(*c*) *If the terms are alternately positive and negative, and if the terms are numerically decreasing with zero as their limit, the instinctive feeling just mentioned in* (*b*) **is actually correct**: the series $a_0-a_1+a_2-a_3+\cdots$ converges if a_n approaches zero; the error made in stopping with a_n is less than a_{n+1}.†

For, the sum $s_n=a_0-a_1+\cdots\pm a_{n-1}$ evidently alternates

* This fallacious instinctive feeling is doubtless actually *used*, and it is responsible for more errors than any other single fallacy. The example here mentioned is certainly neither an unusual nor an artificial example.

† One must, however, make quite sure that the terms actually approach **zero**, not merely that they become rather small; the addition of .0000001 to each term would often have no appreciable effect on the appearance of the first few terms, but it would make any convergent series diverge.

between an increase and a decrease as n increases, and this alternate swinging forward and then backward dies out as n increases, since a_n is precisely the amount of the nth swing.

On each swing s_n passes a point S which it again repasses on the return swing; and its distance from that point is never more than the next swing, — never more than a_{n+1}. Since a_n approaches zero, s_n approaches S, as n becomes infinite.

Thus the series for $\sin x$ is particularly easy to use in calculation: the error made in using $x - x^3/3\,!$ in place of $\sin x$ is certainly less than $x^5/5\,!$. The test of § 134 shows, in fact, that the error $|E_5| < M_5 |x^5/5\,!|$, where $M_5 = 1$.

The similar series for e^x:

$$e^x = 1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \cdots + \frac{x^n}{n\,!} + \cdots$$

is not quite so convenient, since the swings are all in one direction for positive values of x; certainly the error in stopping with any term is greater than the first term omitted. The error can be estimated by § 134, p. 257; thus E_5 (for $x > 0$) is less than $M_5 x^5/5\,!$, where M_5 is the maximum of $f^{\mathrm{v}}(x) = e^x$ between $x = 0$ and $x = x$, *i.e.* e^x; hence $E_5 < e^x x^5/5\,!$. Note that $e^x > 1$ for $x > 0$.

Another means of convincing oneself that the preceding series converges is by comparison with a geometric series with a ratio $x/2$, as in Example 2, p. 272. But this method would require the computation of a vast number of terms, to make sure that the error is small.

(*d*) *A consistently small error in the values of a function may make an enormous error in the values of its derivative.*

Thus the function $y = x - .00001 \sin(100000\,x)$ is very well approximated by the single term $y = x$, — in fact the graphs drawn accurately on any ordinary scale will not show the slightest trace of difference between the two curves. Yet the slope of $y = x$ is always 1, while the slope of $y = x - .00001 \sin(100000\,x)$ varies from 0 to 2 with extreme rapidity. Draw the curves, and find dy/dx for the given function.

One advantage in Taylor series and Taylor approximating polynomials is the known fact — proved in advanced texts — that *differentiation as well as integration is quite reliable on any valid* **Taylor** *approximation.**

* See, *e.g.*, Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 380.

Thus an attempt to expand the function $y = x - .00001 \sin(100000\,x)$ in **Taylor** form gives

$$y = x - \left[x - \frac{100000^2}{3!}x^3 + \frac{100000^4}{5!}x^5 - \cdots \right],$$

which would never be mistaken for $y = x$ by any one; the series indeed converges and represents y for every value of x, but a very hasty examination is sufficient to show that an enormous number of terms would have to be taken to get a reasonable approximation, and no one would try to get the derivative by differentiating a single term.

If the relation expressed by the given equation was obtained by experiment, however, no reliance can be placed in a formal differentiation, even though Taylor approximations are used, for minute experimental errors may cause large errors in the derivative. Attention is called to the fact that the preceding example is not an unnatural one, — precisely such rapid minute vibrations as it contains occur very frequently in nature.

EXERCISES LIX. — INFINITE SERIES

1. Show that the series obtained by long division for $1 \div (1 + x)$ is the same as that given by Taylor's Series.

2. Obtain the series for $\log(1 + x)$ (see Ex. 2 (*d*), List LVIII), by integrating the terms of the series found in Ex. 1 separately.

3. Find the first four terms of the series for $\sin^{-1} x$ in powers of x directly; then also by integration of the separate terms of the series for $1/\sqrt{1-x^2}$.

4. Proceed as in Ex. 3 for the functions $\tan^{-1} x$ and $1/(1 + x^2)$.

5. Show that the series for $\cos x$ in powers of x is obtained by differentiating separately the terms of the series for $\sin x$.

6. Show that repeated differentiation or integration of the separate terms of the series for e^x always results in the same series as the original one.

7. From the series for $\tan^{-1} x$ compute π by using the identity $\pi/4 = 4 \tan^{-1}(1/5) - \tan^{-1}(1/239)$.

8. $S(x) = \int_0^x (\sin u/u)du = x - \frac{1}{3}\frac{x^3}{3!} + \frac{1}{5}\frac{x^5}{5!} - \cdots.$

Show that $S(.1) = .0999+$; $S(1) = .9461$; $S(3) = 1.8487$.

9. The Gudermannian of x is $gd(x) = 2\tan^{-1} e^x - \pi/2$; expand in powers of x; calculate $gd(.1) = 5° \, 43'$, and $gd(.7) = 37° \, 11'$.

10. The Fresnel integrals are

$$C(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\cos z}{\sqrt{z}} dz; \quad S(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\sin z}{\sqrt{z}} dz.$$

Obtain power series in z for $C(z)$ and $S(z)$. Calculate $C(.1) = .2521$, $C(1) = .7217$, $C(3) = .5610$; $S(3) = .7117$, $S(.1) = .0924$, $S(5) = .4659$.

11. The graphs of $C(z)$ and $S(z)$ (Ex. 10) are wave curves of decreasing amplitude. Locate the crests of the waves. Draw the graphs.

12. The "error integral" is $P(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-x^2}\,dx$. Express $P(x)$ as a series in powers of x; calculate $P(.1) = .1125$, $P(1) = .8427$, $P(2) = .9953+$.

13. Show that $\displaystyle\int_0^\infty \sqrt{t}\,e^{-t}\,dt = .8862+$. **15.** Show that $\displaystyle\int_0^{\frac{1}{2}} dt/\sqrt{1-t^3} = .508+$.

14. Show that $\displaystyle\int_0^\infty t^{0.3}e^{-t}\,dt = .8975+$. **16.** Show that $\displaystyle\int_0^1 dt/\sqrt{1-t^4} = 1.311+$.

17. Show that $\displaystyle\int_0^{\pi/2} \sin^{5/4} x\,dx = .9309+$.

18. $K = \displaystyle\int_0^{\pi/2} (1/\sqrt{1-k^2\sin^2\phi})\,d\phi = \frac{\pi}{2}[1 + (1/2)^2k^2 + (1\cdot 3/2\cdot 4)^2k^4 + (1\cdot 3\cdot 5/2\cdot 4\cdot 6)^2k^6 + \cdots]$. Obtain this result. The time of swing of a simple pendulum of length l through an angle α is $4\sqrt{l/g}\,K$, where $k = \sin(\alpha/2)$. Compute this time when $\alpha = 60°$. (See Ex. 15, p. 254; and *Tables*, V, D.)

19. $E = \displaystyle\int_0^{\pi/2} \sqrt{1-k^2\sin^2\phi}\,d\phi = \frac{\pi}{2}[1 - (1/2)^2k^2 - [(1\cdot 3)/(2\cdot 4)]^2(k^4/3) - [(1\cdot 3\cdot 5)/(2\cdot 4\cdot 6)]^2(k^6/5) - \cdots]$. Obtain this result. Show that the perimeter of an ellipse, whose major axis is $2a$ and eccentricity k, is $4aE$. Calculate to 1 % this length when $a = 2$ and $k = 1/2$. (See *Tables*, V, E.)

20. The Bessel Function of order zero is defined by $J_0(x) = 1 - (x/2)^2/(1\,!)^2 + (x/2)^4/(2\,!)^2 - (x/2)^6/(3\,!)^2 + \cdots$.
Calculate: $J_0(.2) = .9900$; $J_0(1) = .7652$; $J_0(3) = -.2601$.
Show that $J_0(x)$ is a solution of the equation $y'' + y'/x - y = 0$.

21. The Bessel Function of order one is $J_1(x) = (x/2)[1 - (x/2)^2/(1\cdot 2) + (x/2)^4/(1\cdot 2\cdot 2\cdot 3) - (x/2)^6/(1\cdot 2\cdot 3\cdot 2\cdot 3\cdot 4) + \cdots]$.
Calculate: $J_1(.2) = .0995$; $J_1(1) = .4401$; $J_1(3) = .3391$.
Show that $J_1(x)$ satisfies the equation $y'' + y'/x + (1 + 1/x^2)y = 0$.

22. In the flow of water through a channel or pipe the "mean hydraulic radius" is defined as "cross section of stream ÷ wetted perimeter." Calculate the m. h. r. for an elliptical pipe flowing full, the axes of the ellipse being 4 in. and 3 in. respectively. (See Ex. 19.) Compare with result for a circular pipe of the same cross section.

CHAPTER IX

SEVERAL VARIABLES PARTIAL DERIVATIVES APPLICATIONS GEOMETRY

PART I. PARTIAL DIFFERENTIATION — ELEMENTARY APPLICATIONS

144. Partial Derivatives. If one quantity depends upon two or more other quantities, its rate of change with respect to one of them, while all the rest remain fixed, is called a **partial derivative.***

If $z = f(x, y)$ is a function of x and y, then, for a constant value of y, $y = k$, z is a function of x alone: $z = f(x, k)$; the derivative of this function of x alone is called the **partial derivative** of z **with respect to** x, and is denoted by any one of the symbols

$$\frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x} = f_x(x, y) = \frac{df(x, k)}{dx}$$
$$= \lim_{\Delta x \doteq 0} \frac{f(x + \Delta x, k) - f(x, k)}{\Delta x}.$$

A precisely similar formula defines the partial derivative of z with respect to y which is denoted by $\partial z / \partial y$.

In general, if u is a function of any number of variables x, y, z, $\cdots$, and if one calculates the first derivative of u with

* This notion is perhaps more prevalent in the world at large than the notion of a derivative of a function of one variable, because quantities in nature usually depend upon a great many influences. The notion of *partial derivative* is what is expressed in the ordinary phrases "the rate at which a quantity changes, everything else being supposed equal," or ". . . other things being the same." The reason for the existence of this idea is the attempt to estimate the effect of each contributing cause apart from that of all others. Compare Ex. 15, p. 254, Ex. 24, p. 90, and many others.

respect to each of these variables, supposing all the others to be fixed, the results are called the first partial derivatives of u with respect to $x, y, z, \cdots$, respectively, and are denoted by the symbols

$$\partial u/\partial x, \quad \partial u/\partial y, \quad \partial u/\partial z, \cdots.$$

145. Technique. No new rules are necessary.

Example 1. Given $z = x^2 + y^2$, to find $\partial z/\partial x$ and $\partial z/\partial y$.

To find $\partial z/\partial x$, think of y as constant: $y = k$; then

$$\frac{\partial z}{\partial x} = \frac{\partial (x^2 + y^2)}{\partial x} = \frac{d(x^2 + k^2)}{dx} = 2x; \quad \frac{\partial z}{\partial y} = 2y.$$

Example 2. Given $z = x^2 \sin(x + y^2)$, to find $\partial z/\partial x$ and $\partial z/\partial y$.

$$\frac{\partial z}{\partial x} = \frac{\partial \{x^2 \sin(x + y^2)\}}{\partial x} = \left[\frac{d\{x^2 \sin(x + k^2)\}}{dx}\right]_{y=k}$$

$$= 2x \sin(x + y^2) + x^2 \cos(x + y^2).$$

$$\frac{\partial z}{\partial y} = \frac{\partial \{x^2 \sin(x + y^2)\}}{\partial y} = \left[\frac{d\{k^2 \sin(k + y^2)\}}{dy}\right]_{x=k}$$

$$= 2x^2 y \cos(x + y^2).$$

146. Higher Partial Derivatives. Successive differentiation is carried out as in the case of ordinary differentiation. There are evidently four ways of getting a **second** partial derivative: differentiating twice with respect to x; once with respect to x, and then once with respect to y; once with respect to y, and then once with respect to x; twice with respect to y. These four second derivatives are denoted, respectively, by the symbols

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y); \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\, \partial x} = f_{xy}(x, y);$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y); \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\, \partial y} = f_{yx}(x, y).$$

There is no new difficulty in carrying out these operations; in fact the situation is simpler than one might suppose, for it turns out that the two cross derivatives f_{xy} and f_{yx} are always equal; the order of differentiation is immaterial.*

A similar notation is used for still higher derivatives:

$$f_{xxx}=\frac{\partial^3 z}{\partial x^3}=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right);\ f_{yxx}=\frac{\partial^3 z}{\partial y\partial x^2}=\frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right);$$

etc.; and the order of differentiation is immaterial.

The **order** of a partial derivative is the total number of successive differentiations performed to obtain it. The partial derivatives of the first and second orders are very frequently represented by the letters p, q, r, s, t:

$$p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y},\ r=\frac{\partial^2 z}{\partial x^2},\ s=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x},\ t=\frac{\partial^2 z}{\partial y^2}.$$

Example 1. Given $z=x^2\sin(x+y^2)$, show that $f_{xy}=f_{yx}$.

Continuing Example 2, § 145, we find:

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left[2x\sin(x+y^2)+x^2\cos(x+y^2)\right]$$

$$=4xy\cos(x+y^2)-2x^2y\sin(x+y^2).$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left[2x^2y\cos(x+y^2)\right]$$

$$=4xy\cos(x+y^2)-2x^2y\sin(x+y^2).$$

EXERCISES LX. — TECHNIQUE OF PARTIAL DIFFERENTIATION

1. The volume of a right circular cylinder is $v=\pi r^2h$. Find the rate of change of the volume with respect to r when h is constant, and express it as a partial derivative. Find $\partial v/\partial h$, and express its meaning.

2. The pressure p, the volume v, and temperature θ of a gas are connected by the relation $pv=k\theta$, where θ is measured from the absolute zero, $-273°$ C. Assuming θ constant, find $\partial p/\partial v$ and express its meaning. If the volume is constant, express the rate of change of pressure with respect to the temperature as a derivative, and find its value.

* At least if the derivatives are themselves continuous. See Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 13.

3. Find the rate of change of the volume of a cone with respect to its height, if the radius is constant; and the rate of change of the volume with respect to the radius, if the height is a constant.

4. Find the first and second partial derivatives, $\partial z/\partial x$, $\partial z/\partial y$, $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x\,\partial y$, $\partial^2 z/\partial y\,\partial x$, and $\partial^2 z/\partial y^2$ for each of the following functions. In each case verify the fact that $\partial^2 z/\partial x\,\partial y = \partial^2 z/\partial y\,\partial x$.

(*a*) $z = x^2 - y^2$.	(*d*) $z = e^{2x+3y}$.	(*g*) $z = (x+y)e^{x^2+y^2}$.
(*b*) $z = x^2 y + xy^2$.	(*e*) $z = \tan^{-1}(y/x)$.	(*h*) $z = (xy - 2y^2)^{3/2}$.
(*c*) $z = \sin(x^2 + y^2)$.	(*f*) $z = e^x \sin y$.	(*i*) $z = \log(x^2+y^2)^{1/2}$.

5. Verify the fact that $z = x^2 - y^2$ satisfies the equation $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$. Show that the same equation is satisfied by 4 (*e*) and 4 (*i*).

[NOTE. An equation which contains partial derivatives is called a **partial differential equation**. (See p. 382.) The particular equation of this exercise is called **Laplace's equation.**]

6. A point moves parallel to the x-axis. What are the rates of change of its polar coördinates with respect to x?

7. Show that the rate of change of the total surface of a right circular cylinder with respect to its altitude is $\partial A/\partial h = 2\pi r$; and that its rate of change with respect to its radius is $\partial A/\partial r = 2\pi h + 4\pi r$.

8. Calculate the rate of change of the hypotenuse of a right triangle relative to a side, the other side being fixed; relative to an angle, the opposite side being fixed.

9. Two sides and the included angle of a parallelogram are a, b, C, respectively. Find the rate of change of the area with respect to each of them, the other two being fixed; the same for the diagonal opposite to C.

10. In a steady electric current $C = V \div R$, where C, V, R, denote the current, the voltage (electric pressure), and the resistance, respectively. Find $\partial C/\partial V$ and $\partial C/\partial R$, and express the meaning of each of them.

147. Geometric Interpretation. The first partial derivatives of a function of two independent variables

$$z = f(x, y)$$

can be interpreted geometrically in a simple manner. This equation represents a surface, which may be plotted by erect-

ing at each point of the xy-plane a perpendicular of length $f(x, y)$; the upper ends* of these perpendiculars are the points of the surface.

Let $ABCD$ be a portion of this surface lying above an area $abcd$ of the xy-plane. If x varies while y remains fixed, say equal to k, there is traced on the surface the curve HK, the section of the surface by the plane $y = k$. The slope of this curve is $\partial z/\partial x$.

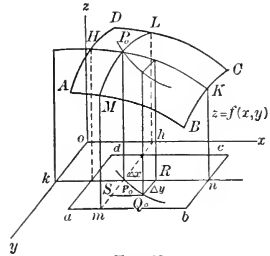


FIG. 63

Similarly, $\partial z/\partial y$ is the slope of the curve cut from the surface by a plane $x = h$.

148. Total Derivative. If in addition to the function $z = f(x, y)$, *a relation between x and y, say $y = \phi(x)$, is given, z reduces by simple substitution to a function of one variable:*

$$z = f(x, y),\ y = \phi(x) \quad \text{gives} \quad z = f(x, \phi(x)).$$

Now any change Δx in x *forces* a change Δy in y; hence y *cannot* remain constant (unless, indeed, $\phi(x) = \text{const.}$). Hence the change Δz in the value of z is due both to the direct change Δx in x and also to the *forced* change Δy in y. We shall call

$\Delta z =$ the total change in $z = f(x + \Delta x, y + \Delta y) - f(x, y)$,

$\Delta_x z =$ the partial change due to Δx directly

$= f(x + \Delta x, y) - f(x, y)$,

$\Delta_y z =$ the partial change *forced* by the *forced* change Δy

$= \Delta z - \Delta_x z, \ = f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)$.

* If z is negative, of course the lower end is the one to take.

It follows that

$$(1)\quad \frac{dz}{dx}=\lim_{\Delta x\doteq 0}\frac{\Delta z}{\Delta x}\qquad\left\{=\lim_{\Delta x\doteq 0}\left(\frac{f(x+\Delta x,\,y+\Delta y)-f(x,\,y)}{\Delta x}\right)\right\};$$

$$=\lim_{\Delta x\doteq 0}\frac{\Delta_x z+\Delta_y z}{\Delta x}\qquad\left\{=\lim_{\substack{\Delta x\doteq 0\\ \Delta y\doteq 0}}\left(\frac{f(x+\Delta x,\,y)-f(x,\,y)}{\Delta x}+\frac{f(x+\Delta x,\,y+\Delta y)-f(x+\Delta x,\,y)}{\Delta y}\frac{\Delta y}{\Delta x}\right)\right\};$$

whence, if the partial derivatives exist and are continuous,*

$$(2)\qquad \frac{dz}{dx}=\lim_{\Delta x\doteq 0}\left[\frac{\Delta_x z}{\Delta x}+\frac{\Delta_y z}{\Delta y}\frac{\Delta y}{\Delta x}\right]=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\frac{dy}{dx},$$

or, multiplying both sides by $dx(=\Delta x)$,

$$(3)\qquad dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy,\ \text{since } dy=\frac{dy}{dx}dx,$$

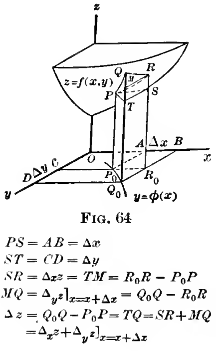


FIG. 64

$PS = AB = \Delta x$
$ST = CD = \Delta y$
$SR = \Delta_x z = TM = R_0R - P_0P$
$MQ = \Delta_y z]_{x=x+\Delta x} = Q_0Q - R_0R$
$\Delta z = Q_0Q - P_0P = TQ = SR + MQ$
$= \Delta_x z + \Delta_y z]_{x=x+\Delta x}$

where $dy = \phi'(x)\,dx$. Since $\phi(x)$ is any function whatever, dy is really perfectly arbitrary. Hence (3) holds for any arbitrary values of dx and dy whatever, where $dz = (dz/dx)\,dx$ is defined by (2); dz is called the **total differential of** z.

These quantities are all represented in the figure geometrically: thus $\Delta z = \Delta_x z + \Delta_y z$ is represented by the geometrical equation $TQ = SR + MQ$. It should be noticed that dz is the height of the plane drawn tangent to the surface at P, since

* For a more detailed proof using the law of the mean, see Goursat-Hedrick, *Mathematical Analysis*, I, pp. 38–42.

dz/dx and dz/dy are the *slopes* of the sections of the surface by $y = y_P$ and $x = x_P$, respectively. [See also § 164, p. 321.]

If the curve P_0Q_0 in the xy-plane is given in parameter form, $x = \phi(t)$, $y = \psi(t)$, we may divide both sides of (3) by dt and write

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}, \tag{4}$$

since $dx \div dt = dx/dt$, $dy \div dt = dy/dt$.

149. Elementary Use. In elementary cases, many of which have been dealt with successfully before § 148, the use of the formulas (2), (3), and (4) of § 148 is quite self-evident.

Example 1. The area of a cylindrical cup with no top is

$$A = 2\pi rh + \pi r^2, \tag{1}$$

where h is the height, and r is the radius of the base. If the volume of the cup, $\pi r^2 h$, is known in advance, say $\pi r^2 h = 10$ (cubic inches), we actually do know a relation between h and r:

$$h = \frac{10}{\pi r^2}, \tag{2}$$

whence

$$A = 2\pi r \frac{10}{\pi r^2} + \pi r^2 = \frac{20}{r} + \pi r^2 \tag{3}$$

from which dA/dr can be found. We did precisely the same work in Ex. 7, p. 68. In fact even then we might have used (1) instead of (3), and we might have written

$$\frac{dA}{dr} = 2\pi r\frac{dh}{dr} + 2\pi h + 2\pi r, \text{ or } dA = 2\pi r\, dh + (2\pi h + 2\pi r)\, dr, \tag{4}$$

where dh/dr is to be found from (2).

This is precisely what formula (2), § 148, does for us; for

$$\frac{\partial A}{\partial r} = 2\pi h + 2\pi r, \quad \frac{\partial A}{\partial h} = 2\pi r, \tag{5}$$

$$\frac{dA}{dr} = (2\pi h + 2\pi r) + (2\pi r)\frac{dh}{dr}, \text{ or } dA = (2\pi h + 2\pi r)\, dr + 2\pi r\, dh.$$

We used just such equations as (4) to get the critical values in finding extremes for $dA/dr = 0$ at a critical point. We may now use (2), § 148, to find dA/dr; and the work is considerably shortened in some cases.

Example 2. The derivative dy/dx can be found from (2), § 148, if we know that z is constant.

Thus in § 26, p. 44, we had the equation

$$x^2 + y^2 = 1, \tag{1}$$

and we wrote:

$$\frac{d(x^2+y^2)}{dx} = 2x + 2y\frac{dy}{dx} = \frac{d(1)}{dx} = 0, \tag{2}$$

whence we found

$$x + y\frac{dy}{dx} = 0, \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}. \tag{3}$$

This work may be thought of as follows:

Let $z = x^2 + y^2$; then

$$\frac{dz}{dx} = \frac{d(x^2+y^2)}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 2x + 2y\frac{dy}{dx};$$

but $z = 1$ by (1) above; hence $dz/dx = 0$, and

$$2x + 2y\frac{dy}{dx} = 0, \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Thus the use of the formulas of § 148 is essentially **not at all new**; the preceding exercises and the work we have done in §§ 26, 34, etc., really employ the same principle. But the same facts appear in a new light by means of § 148; and the new formulas are a real assistance in many examples.

150. Small Errors. Partial Differentials. Another application closely allied to the work of § 132, p. 252, is found in the estimation of small errors.

Example 1. The angle A of a right triangle ABC ($C = 90^\circ$), may be computed by the formula

$$\tan A = \frac{a}{b}, \quad \text{or} \quad A = \tan^{-1}\frac{a}{b},$$

where a, b, c are the sides opposite A, B, C. If an error is made in measuring a or b, the computed value of A is of course false. We may estimate the error in A caused by an error in measuring a, supposing temporarily that b is correct, by § 132; this gives approximately

$$\Delta_a A = \frac{\partial A}{\partial a}\Delta a = \frac{\frac{1}{b}}{1+\frac{a^2}{b^2}}\Delta a = \frac{b}{a^2+b^2}\Delta a,$$

where ∂ is used in place of d of § 132, since A really depends on b also, and we have simply supposed b constant temporarily. Likewise the error in A caused by an error in b is approximately,

$$\Delta_b A = \frac{\partial A}{\partial b}\Delta b = \frac{-\frac{a}{b^2}}{1+\frac{a^2}{b^2}}\Delta b = \frac{-a}{a^2+b^2}\Delta b.$$

If errors are possible in both measurements, the total error in A is, approximately, the sum of these two partial errors:

$$|\Delta A| \leqq |\Delta_a A| + |\Delta_b A| = \frac{b\,|\Delta a| + a\,|\Delta b|}{a^2+b^2}.$$

The methods of § 133, p. 256, give a means of finding how nearly correct these estimates of $\Delta_a A$, $\Delta_b A$, and ΔA are; in practice, such values as those just found serve as a guide, since it is usually desired only to give a general idea of the amounts of such errors.

This method is perfectly general. The differences in the value of a function $z = f(x, y)$ of two variables, x and y, which are caused by differences in the value of x alone, or of y alone, are denoted by $\Delta_x z$, $\Delta_y z$, respectively. The total difference in z caused by a change in both x and y is

$$\begin{aligned}\Delta z &= f(x+\Delta x, y+\Delta y) - f(x, y)\\ &= [f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y)] + [f(x+\Delta x, y) - f(x, y)]\\ &= \Delta_y z]_{x=x+\Delta x} + \Delta_x z,\end{aligned}$$

as in § 148. The differences $\Delta_x z$ and $\Delta_y z$ are, approximately,*

* More precisely, these errors are:

$$\Delta_x z = \frac{\partial z}{\partial x}\cdot \Delta x + E'_2, \quad \Delta_y z = \frac{\partial z}{\partial y}\bigg]_{x=x+\Delta x}\cdot \Delta y + E''_2,$$

where $|E'_2|$ and $|E''_2|$ are less than the maximum M_2 of the values of all of the second derivatives of z near (x, y) multiplied by $\overline{\Delta x}^2$, or $\overline{\Delta y}^2$, respectively (see § 133). And since $\partial z/\partial y$ is itself supposed to be continuous, we may write

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + E_2,$$

where $|E_2|$ is less than $M_2(|\Delta x| + |\Delta y|)^2$. [Law of the Mean. Compare § 133.]

$$\Delta_x z = \frac{\partial z}{\partial x}\Delta x, \quad \Delta_y z = \frac{\partial z}{\partial y}\Delta y;$$

whence, approximately,

$$\Delta z = \Delta_y z + \Delta_x z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y.$$

The products $(\partial z/\partial x)\,dx$ and $(\partial z/\partial y)\,dy$ are often called the **partial differentials** of z, and are denoted by

$$\partial_x z = \frac{\partial z}{\partial x}dx, \ \partial_y z = \frac{\partial z}{\partial y}dy, \text{ whence } dz = \partial_x z + \partial_y z,$$

where $dx = \Delta x$ and $dy = (dy/dx)\Delta x = \Delta y$, approximately. We have therefore, approximately,

$$\Delta z = \partial_x z + \partial_y z,$$

within an amount which can be estimated as in § 133 and in the preceding footnote.

Similar formulas give an estimate of the values of the changes in a function $u = f(x, y, z)$ of the variables x, y, z; we have, approximately,

$$\Delta_x u = \frac{\partial u}{\partial x}\Delta x, \ \Delta_y u = \frac{\partial u}{\partial y}\Delta y, \quad \Delta_z u = \frac{\partial u}{\partial z}\Delta z,$$

$$\Delta u = \Delta_x u + \Delta_y u + \Delta_z u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z,$$

within an amount which can be estimated as in the preceding footnote. The generalization to the case of more than three variables is obvious.

EXERCISES LXI. — TOTAL DERIVATIVES AND DIFFERENTIALS

1. Express the total surface area A of a cylindrical can with a bottom but no top, in terms of the height h and the radius of the base r. If the volume of the can is given, say 100 cu. in., find a relation between h and r; and find dA/dr.

2. Find the most economical dimensions for the can described in Ex. 1.

3. Find the most economical dimensions for a funnel made in the form of a right cone, neglecting the outlet hole.

4. The pressure p, the volume v, and the temperature θ of any gas are connected by the relation $pv = k\theta$, when k is a constant. When no heat escapes or enters it is found by experiment that $p = c \cdot v^{-1.41}$ for air. Express θ in terms of v alone and find $d\theta/dv$. Find the same result directly by § 149.

5. Find dz when z is given in terms of x and y, and y is given in terms of x, by one of the following sets of equations:

(a) $z = x^2 + y^2, \quad y = 2x + 3.$

(b) $z = x^2 - y^2, \quad y = x^{3/2}.$

(c) $z = x^2 - y^2, \quad y = x.$

(d) $z = xy, \quad y = \sqrt{2x + 3}.$

(e) $z = \sin(x + y), \quad y = x.$

(f) $z = \sqrt{x^2 + y^2}, \quad y = 1/x.$

6. Find dz/dt when $z = xy$, and $x = \sin t$, $y = \cos t$ by expressing z in terms of t; without expressing z in terms of t. Interpret this result geometrically.

7. Find dy/dx in each of the following implicit equations by method of Ex. 2, § 149:

(a) $x^2 + 4y^2 = 1.$

(b) $4x^2 - 9y^2 = 36.$

(c) $x^3 + y^3 - 3xy = 0.$

(d) $y^2(2a - x) = x^3.$

8. If A, B, C denote the angles, and a, b, c the sides opposite them, respectively, in a plane triangle, and if a, A, B are known by measurements, $b = a \sin B/\sin A$. Show that the error in the computed value of b due to an error da in measuring a is, approximately,

$$\partial_a b = \sin B \csc A \, da.$$

Likewise show that

$$\partial_A b = -a \sin B \csc A \operatorname{ctn} A \, dA, \text{ and } \partial_B b = a \cos B \csc A \, dB;$$

and the total error is, approximately, $db = \partial_a b + \partial_A b + \partial_B b$. Note that A and B are expressed in radian measure.

9. The measured parts of a triangle and their probable errors are

$$a = 100 \pm .01 \text{ ft.}, \quad A = 100^\circ \pm 1', \quad B = 40^\circ \pm 1'.$$

Show that the partial errors in the side b are

$$\partial_a b = \pm .007 \text{ ft.}, \quad \partial_A b = \pm .003 \text{ ft.}, \quad \partial_B b = \pm .023 \text{ ft.}$$

If these should all combine with like signs, the maximum total error would be

$$db = \pm .033 \text{ ft.}$$

10. If $a = 100$ ft., $B = 40^\circ$, $A = 100^\circ$, and each is subject to an error of 1%, find the per cent of error in b.

11. Find the partial and total errors in angle B, when

$$a = 100 \pm .01 \text{ ft.}, \quad b = 159 \pm .01 \text{ ft.}, \quad A = 30^\circ \pm 1'.$$

12. The radius of the base and the altitude of a right circular cone being measured to 1%, what is the possible per cent of error in the volume? *Ans.* 3%.

13. The formula for index of refraction is $m = \sin i/\sin r$, i being the angle of incidence and r the angle of refraction. If $i = 50^\circ$ and $r = 40^\circ$, each subject to an error of 1%, what is m, and what its actual and its percentage error?

14. Water is flowing through a pipe of length L ft., and diameter D ft., under a head of H ft. The flow, in cubic feet per minute, is $Q = 2356\sqrt{\dfrac{HD^5}{L + 30\,D}}$. If $L = 1000$, $D = 2$, and $H = 100$, determine the change in Q due to an increase of 1% in H; in L; in D. Compare the partial differentials with the partial increments.

15. If the coördinates (x, y) are changed to polar coördinates (ρ, θ), find z in terms of ρ and θ if $z = 4\,x^2 + y^2$; find $\partial z/\partial \rho$ and $\partial z/\partial \theta$.

16. Find $\partial z/\partial \rho$ and $\partial z/\partial \theta$ if $z = xy - 3\,y^2$.

17. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = \rho^2 - 2\,\rho\cos\theta$, where (ρ, θ) are the polar coördinates of the point (x, y).

18. Find $dz/d\theta$ if $z = x^2 - 4\,y^2$, where $x = a\tan\theta$, $y = a\sec\theta$, by expressing z in terms of θ; without expressing z in terms of θ.

19. Find dz/dt if $z = e^{x^2+y^2}\sin(x^2+y)$, where $x = 1 + 2\,t + t^2$, $y = te^{-t}$.

151. Significance of Partial and Total Derivatives. The formulas of § 148 become of vital importance in scientific and mathematical problems. The methods employed are illustrated by the following typical examples.

Example 1. *Expansion of a Gas at Constant Temperature.** Thus in the case of a gas under pressure p, we have

$$(1) \qquad pv = k\theta,$$

* Often called *isothermal* expansion.

where v is the volume and θ is the *absolute* temperature, that is, $\theta = C + 273°$ where C is the temperature (C.). In general, we have

$$(2) \qquad \frac{d\theta}{dv} = \frac{\partial\theta}{\partial v} + \frac{\partial\theta}{\partial p}\frac{dp}{dv} = \frac{p}{k} + \frac{v}{k}\frac{dp}{dv}.$$

If the temperature is constant during a change in volume, the pressure must change, for $d\theta/dv = 0$, and therefore

$$(3) \qquad \frac{\partial\theta}{\partial p}\frac{dp}{dv} + \frac{d\theta}{dv} = 0, \quad \text{or} \quad \frac{dp}{dv} = -\frac{\partial\theta/\partial v}{\partial\theta/\partial p} = -\frac{p}{v},$$

where $\partial\theta/\partial v = p/k$ is the rate of change of temperature which would occur if v alone were changed.

Here again the bare fact that $dp/dv = -p/v$ can be obtained without using § 148. For since θ is constant, $pv=$const., hence $p\,dv + v\,dp = 0$ and $dp/dv = -p/v$. (See Ex. 26, p. 90.)

The new fact discovered by § 148 is the *second* equation in (3), which says that the rate of change of pressure with respect to volume in expansion at constant temperature, is equal to the negative of the ratio of the rate of change of temperature when the volume alone changes to the rate of change of temperature when the pressure alone changes. This fact remains strictly true even when (1) is not strictly true; for if the temperature can be expressed as any function of the volume and the pressure, the first equation under (3) remains true. (See Ex. 28, p. 57.)

*Example 2. Expansion when no Heat escapes or enters.** The fundamental equation $pv = k\theta$ of Ex. 1 and equation (2) hold true in any case for perfect gases. If no heat escapes from nor enters the gas, its temperature is bound to rise or fall if the product pv of the pressure and the volume does not remain constant; for $d\theta/dv = 0$ if and only if $dp/dv = -p/v$.

For any gas the application of sudden mechanical pressure — such as that of the piston of an air compressor — results in some actual decrease in volume, but the rate dp/dv depends on the nature of the particular gas. For air, it is found experimentally that $pv^{1.41} =$ const. (nearly); whence, from (1),

$$\theta = \frac{vp}{k} = \frac{cv^{-0.41}}{k}, \quad \frac{dp}{dv} = -1.41\,c \cdot v^{-2.41}, \quad \frac{d\theta}{dv} = -0.41\,\frac{c}{k}v^{-1.41}.$$

Using (2), we might have written

$$\frac{d\theta}{dv} = \frac{p}{k} + \frac{v}{k}\frac{dp}{dv} = \frac{cv^{-1.41}}{k} + \frac{v}{k}(-1.41\,cv^{-2.41}) = -0.41\,\frac{c}{k}v^{-1.41}.$$

* Often called *adiabatic* expansion.

These two results for $d\theta/dv$, found from (1) and from (2) of Ex. 1, must of course agree.

Example 3. *Implicit Equations. Contour Lines on a Surface.* If the equation of a plane curve is given in implicit form,

(1) $$f(x, y) = 0,$$

we may think of the curve as a section of the surface

(2) $$z = f(x, y)$$

by the plane $z = 0$ (compare Ex. 2, p. 288). Then (2), § 148, becomes

(3) $$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 0,$$

since $dz/dx = d0/dx = 0$. Hence, solving for dy/dx,

(4) $$\frac{dy}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y}.$$

The same equation holds for any section of the surface by any horizontal plane $z = k$. Such a section is often called a **contour line**.

Notice that the value of dy/dx given by (4) is equivalent to the value found by the method of § 26, p. 44. In that paragraph, a value of dy/dx was discarded if the point (x, y) did not lie on the given curve (1). It is easy to see now that in any case (4) expresses the value of dy/dx at *any* point (x, y) for the particular *contour line* through that point.

Example 4. *Flow of Heat in a Metal Plate. Directional Derivative.* Let us suppose that a metal plate is steadily warmed on one edge (*e.g.* by a gas burner) and steadily cooled on the other (*e.g.* by a water jacket). Then, after a lapse of time, the temperature at every point in the plate is quite fixed, though the temperature is different at different points. Thus, the temperature θ at any point (x, y) is a function of x and of y,

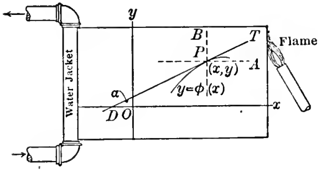


FIG. 65

(1) $$\theta = f(x, y).$$

Let $y = \phi(x)$ be any curve through a point P; if we follow the variations in temperature *along that curve*,

(2) $$\theta = f(x, y),\ y = \phi(x),$$

we have, by (3), § 148,

$$(3) \qquad d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy.$$

Let s be the length of the arc of the curve; then $ds^2 = dx^2 + dy^2$, $dx/ds = \cos\alpha$, $dy/ds = \sin\alpha$, where $\alpha = \angle xDT$, by § 62, p. 107; hence, dividing both sides of (3) by ds, we have

$$(4) \qquad \frac{d\theta}{ds} = \frac{\partial \theta}{\partial x}\frac{dx}{ds} + \frac{\partial \theta}{\partial y}\frac{dy}{ds} = \frac{\partial \theta}{\partial x}\cos\alpha + \frac{\partial \theta}{\partial y}\sin\alpha.$$

This equation shows that the rate of change of the temperature along the curve depends only on the angle α; all curves tangent to DPT at P give the same result for $d\theta/ds$.

The equation (4) evidently holds for any function $\theta = f(x, y)$ whatever; often the derivative $d\theta/ds$ is called a *directional derivative*, that is, a derivative (or rate of change) of θ in the direction PT.

Example 5. Flow of Water in Pipes. Particle Derivative. When water is flowing in a pipe, the speed of the water may be considered as follows:

(*a*) Fixing our attention upon a particular point A in the pipe, — say its mouth, — we may consider the speed of various water particles which pass that point. This speed S_A may change as time goes on, if the water pressure varies from any cause.

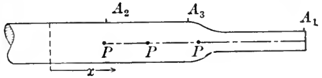


FIG. 66

(*b*) Fixing our attention on a particular *water particle* P, as it moves through the pipe, that particular particle has a speed, S_P, which may change even when the flow through the pipe is perfectly constant; for if P moves from a wide part of the pipe to a comparatively narrow part (as in the nozzle of a hose) the speed S_P increases. It is clear that $S_P = S_A$, when P is at A.

(*c*) Let us suppose the pressure is constant. Then S_A depends only on the (fixed) position of the point A; but S_P depends upon the time t, since the position of P changes with the time:

$$\frac{\partial S_A}{\partial t} = 0, \quad \frac{\partial S_P}{\partial t} \neq 0.$$

(*d*) If the *water pressure* changes, both S_P and S_A change; that is, S_A and S_P both depend on the pressure p:

$$S_A = \text{a function of } p \text{ alone};$$
$$S_P = \text{a function of } p \text{ and of } t.$$

If t is assigned a fixed value, $S_P\Big]_{t=k} = S_A$, where A is the position of P at the time $t = k$. We have

$$\frac{dS_P}{dt} = \frac{\partial S_P}{\partial t} + \frac{\partial S_P}{\partial p}\frac{dp}{dt},$$

where $\partial S_P/\partial p$ is the rate of change in S_P with respect to p which would occur if t alone could be kept constant, *i.e.* the rate of change of S_A with respect to p, or dS_A/dp; and $\partial S_P/\partial t$ is the rate of change of S_P with respect to t which would exist if p alone were constant.* The equation therefore shows that the actual rate of change of S_P with respect to t is equal to the rate at which S_P would change if p were constant plus the rate at which S_A changes with respect to p times the rate of change of p with respect to t. All of these concepts can be illustrated by the speeds of water particles near the nozzle of an ordinary garden hose as the water is turned on or off.

EXERCISES LXII. — APPLICATIONS OF TOTAL DERIVATIVES

1. Find dz when $z = x^2 + 4y^2$. Hence find dy/dx for the point $x = 1$, $y = 2$ on the curve $x^2 + 4y^2 = 17$. Find dy/dx for that curve of the family $x^2 + 4y^2 = k$ which passes through $x = 2$, $y = 1$, at that point.

2. Find dy/dx for that curve of the family $xy = k$ which passes through the point $x = 2$, $y = 3$, at that point.

3. Find dy/dx for that curve of the family $x^3 + y^3 - 3xy = k$ which passes through the point $(1, 1)$ at that point.

4. For steam, it is found by experiment that $pv^{17/16} = \text{const.}$, for adiabatic expansion. Find dp/dv and $d\theta/dv$, where θ, v, p denote the temperature, volume, and pressure, respectively, and $pv = k\theta$.

5. The strength of a beam is proportional to bd^2/l, where b is the breadth, d the depth, and l the length of the beam. Discuss the effect upon the strength of changes in each dimension separately; the effect of simultaneous changes in b and d when l is constant.

6. In the beam of Ex. 5 if b and d are changed while l is constant, find a relation connecting b and d if the strength remains unchanged. Find the rate of change of b with respect to d under these circumstances.

* For this reason, the partial derivative $\partial S_P/\partial t$ is often called a "*particle*" derivative: it is in this case the fictitious rate at which S_P would change if the flow *were* steady, as P moves along the pipe. The other derivative $\partial S_P/\partial p$ may be replaced by dS_A/dp.

7. The amount of deflection D of a rectangular beam under a load is proportional to l^3/bd^3, in the notation of Ex. 5. Find the rates of change of D with respect to each dimension separately. Find a relation between l and d for which D is constant while b is constant; and find dl/dd in this case. Find a relation between l and b for which D and d are constant and find db/dl in this case.

8. For the beam of Ex. 7 show that if b is constant,

$$dD = (3\,l^2/bd^3)[dl - (l/d)dd].$$

9. The collapsing pressure of a boiler tube is given by Fairburn as proportional to t^2/ld where t, l, d, respectively, denote the thickness of the material, the length, and the diameter, of the tube. Show how the collapsing pressure changes with respect to changes in t and d.

10. The resistance R, due to water friction for a boat in still water, is proportional to $S^2D^{2/3}$, where S is the speed and D is the displacement. Show how the resistance changes when S changes; when D changes.

11. If the boat of Ex. 10 is loaded more heavily, D increases; but S is usually decreased. Find dS/dD if R is kept constant.

12. The temperature at points of a certain square plate $OABC$ varies inversely as $1 + r^2$, where r is the distance from O. The temperature at O is 100°. Find the rate of change of the temperature (a) along the diagonal OB, (b) along AC at the center of the plate; (c) along a vertical line through the center of the plate.

13. If u is a function of three variables, such as the density or the temperature at points of a solid, the rates of change of u in the directions of the coördinate axes are, respectively, $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$.

If s is a variable distance along a line making angles α, β, γ with the coördinate axes, show that the rate of variation of u along this line is

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} + \frac{\partial u}{\partial z}\frac{dz}{ds} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma.$$

14. In a spherical shell of inner radius 5 and outer radius 10, the temperature decreases uniformly from 100° at the inner surface to 0° at the outer. Show that the rate of variation of the temperature along a radius, at right angles to a radius, along a line inclined 45° to a radius at their point of intersection are, respectively, -20, 0, $-10\sqrt{2}$.

15. From the value of dy/dx found in Example 3, § 151, show that the equation of the tangent to a plane curve whose equation is given in implicit form, $f(x, y) = 0$, is $(x - x_P)(\partial f/\partial x)_P + (y - y_P)(\partial f/\partial y)_P = 0$, where (x_P, y_P) is the point of tangency, and where the values of the derivatives are to be taken at that point.

PART II. APPLICATIONS TO PLANE GEOMETRY

152. Envelopes. The straight line

(1) $$y = kx - k^2,$$

where k is a constant to which various values may be assigned, has a different position for each value of k. All the straight lines which (1) represents *may* be tangents to some one curve. If they are, the point P_k, (x, y) at which (1) is tangent to the curve, evidently depends on the value of k:

(2) $$x = \phi(k), \quad y = \psi(k);$$

these equations may be considered to be the parameter equations of the required curve. The motive is to find the functions $\phi(k)$ and $\psi(k)$ if possible.

Since P_k *lies on* (1) *and on* (2), we may substitute from (2) in (1) to obtain:

(3) $$\psi(k) = k\phi(k) - k^2,$$

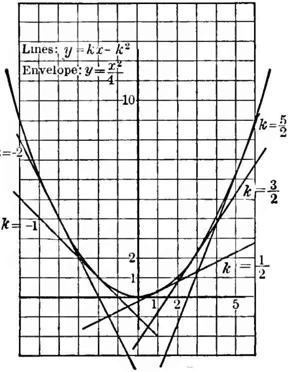


FIG. 67

which must hold for all values of k. Moreover, *since* (1) *is tangent to* (2) *at* P_k, the values of dy/dx found from (1) and from (2) must coincide:

(4) $$k = \left.\frac{dy}{dx}\right]_{\text{from (1)}} = \left.\frac{dy}{dx}\right]_{\text{from (2)}} = \frac{\psi'(k)}{\phi'(k)}, \text{ or } k\phi'(k) = \psi'(k).$$

To find $\phi(k)$ and $\psi(k)$ from the two equations (3) and (4), it is evident that it is expedient to differentiate both sides of (3) with respect to k:

(3*) $$\psi'(k) = k\phi'(k) + \phi(k) - 2k;$$

this equation reduces by means of (4) to the form

(5) $$0 = 0 + \phi(k) - 2k, \text{ or } \phi(k) = 2k,$$

and then (3) gives

(6) $$\psi(k) = k(2k) - k^2 = k^2.$$

Hence the parameter equations (2) of the desired curve are

(7) $$x = 2k, \quad y = k^2,$$

and the equation in usual form results by elimination of k:

(8) $$y = \frac{x^2}{4}.$$

It is easy to show that the tangents to (8) are precisely the straight lines (1)

The preceding method is perfectly general. Given any set of curves

(1)′ $$F(x, y, k) = 0,$$

where k may have various values, a curve to which they are all tangent is called their **envelope**; its equations may be written

(2)′ $$x = \phi(k),\ y = \psi(k);$$

whence by substitution in (1)′,

(3)′ $$F[\phi(k),\ \psi(k),\ k] = 0,$$

for all values of k. Differentiating (3)′ with respect to k,

(3 *)′ $$\frac{dF(x, y, k)}{dk} = \frac{\partial F}{\partial x}\frac{dx}{dk} + \frac{\partial F}{\partial y}\frac{dy}{dk} + \frac{\partial F}{\partial k} = 0.$$

Moreover, since (1)′ is *tangent* to (2)′,

(4)′ $$-\frac{\partial F}{\partial x} \div \frac{\partial F}{\partial y} = \left.\frac{dy}{dx}\right]_{\text{from (1)}'} = \left.\frac{dy}{dx}\right]_{\text{from (2)}'} = \frac{dy}{dk} \div \frac{dx}{dk};$$

whence (3 *)′ reduces to the form

(5)′ $$\frac{\partial F}{\partial k} = 0;$$

and then (3)′ and (5)′ may be solved as simultaneous equations to find $\phi(k)$ and $\psi(k)$ as in the preceding example.

The envelope may be found speedily by simply writing down the equations $(1)'$ and $(5)'$, and then eliminating k between them. It is recommended very strongly that this should not be done until the student is familiar with the direct solution as shown in the preceding example.

153. Envelope of Normals. Evolute. The normal to the curve $y = x^2$ at a point $x = k$ is

$$(1) \qquad y - k^2 = -\frac{1}{2k}(x - k).$$

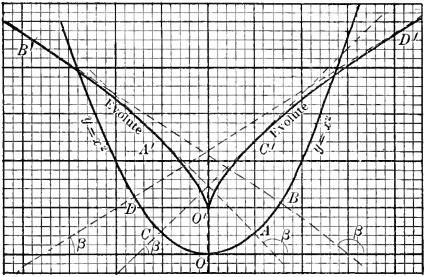


FIG. 68

If these lines, for all values of k, are tangent to some one curve:

$$(2) \qquad x = \phi(k), \quad y = \psi(k),$$

direct substitution gives

$$(3) \qquad \psi(k) - k^2 = -\frac{1}{2k}[\phi(k) - k],$$

for all values of k. Differentiation with respect to k gives *

$$(3^*) \qquad \psi'(k) - 2k = \frac{1}{2k^2}[\phi(k) - k] - \frac{1}{2k}[\phi'(k) - 1].$$

* The reason for this differentiation is brought out in the example of § 152. Notice that the equations here are numbered to correspond exactly to the equations of § 152.

Moreover, since (1) is tangent to (2)

$$(4)\qquad -\frac{1}{2k}=\left.\frac{dy}{dx}\right]_{\text{from }(1)}=\left.\frac{dy}{dx}\right]_{\text{from }(2)}=\psi'(k)\div\phi'(k);$$

whence (3*) reduces to

$$(5)\qquad 0-2k=\frac{1}{2k^2}[\phi(k)-k]-\frac{1}{2k}(0-1).$$

Solving for $\phi(k)$, we find:

$$(6)\qquad \phi(k)=k+2k^2\left(-2k-\frac{1}{2k}\right)=-4k^3.$$

It follows from (3) that

$$(7)\qquad \psi(k)=k^2-\frac{1}{2k}[\phi(k)-k]=3k^2+1/2,$$

whence the equations of the new curve are

$$(8)\qquad x=-4k^3,\quad y=3k^2+1/2.$$

We might proceed to eliminate k, as in the example of § 152, in order to express the equation of the new curve in usual form; but when the elimination is at all difficult, as it is here, it is best to keep the equation in the parameter form (8). The graph may be plotted from these equations as usual. The accurate construction of a few normals to the given curve is a great assistance in drawing this graph.

The new curve is called the **evolute** of the given curve; the given curve is called an **involute** of the new one: *the evolute to any curve is the envelope of its normals.* (See § 154 and Ex. 4, p. 172.)

The method used above is perfectly general, and may be used in any problem. Thus the normal to a curve $y=f(x)$ at a point $x=k$ is

$$(1)'\qquad y-f(k)=-\frac{1}{f'(k)}(x-k).$$

If these normals, for all values of k, are *tangent* to the new curve

$$(2)'\qquad x=\phi(k),\quad y=\psi(k),$$

direct substitution, followed by differentiation with respect to k, gives

$$(3)' \qquad \psi(k) - f(k) = -\frac{1}{f'(k)}[\phi(k) - k],$$

$$(3^*)' \qquad \psi'(k) - f'(k) = \frac{f''(k)}{[f'(k)]^2}[\phi(k) - k] - \frac{1}{f'(k)}[\phi'(k) - 1].$$

Moreover, since $(1)'$ is tangent to $(2)'$

$$(4)' \qquad -\frac{1}{f'(k)} = \left.\frac{dy}{dx}\right]_{\text{from }(1)'} = \left.\frac{dy}{dx}\right]_{\text{from }(2)'} = \psi'(k) \div \phi'(k) ;$$

whence $(3^*)'$ reduces to

$$(5)' \qquad 0 - f'(k) = + \frac{f''(k)}{[f'(k)]^2}[\phi(k) - k] - \frac{1}{f'(k)}[0 - 1] ;$$

an equation which might have been found directly from $(5)'$ of § 152. Solving $(5)'$ for $\phi(k)$, we find: *

$$(6)' \qquad \phi(k) = k + \frac{[f'(x)]^2}{f''(k)}\left[-f'(k) - \frac{1}{f'(k)}\right]$$

$$= k - \frac{f'(k)\{1 + [f'(k)]^2\}}{f''(k)};$$

whence, from $(3)'$

$$(7)' \qquad \psi(k) = f(k) - \frac{1}{f'(k)}[\phi(k) - k] = f(k) + \frac{1 + [f'(k)]^2}{f''(k)}.$$

Denoting $f(k)$ by y_k, $f'(k)$ by m_k (the slope at $x = k$), $f''(k)$ by b_k (the flexion at $x = k$), **the equations of the evolute** may be written in the form:

$$(8)' \qquad x = k - \frac{m_k(1 + m_k^2)}{b_k}, \text{ and } y = y_k + \frac{1 + m_k^2}{b_k}.$$

These equations may be used to write down the equations of the evolute directly; but it is strongly recommended that the direct solution, as above, be practiced. Frequently the elimination of k between the two equations $(8)'$ is rather difficult, as in the example given above; hence the equations are very often left in the parameter form $(8)'$.

* Notice that the work breaks down at this point if $f''(k) = 0$. The advantage of this direct solution is that such special cases are not so troublesome as when the final formulas alone are used.

EXERCISES LXIII. — ENVELOPES EVOLUTES

1. Show that the envelope of the set of straight lines $y = 3kx - k^3$ is $y^2 = 4x^3$.

2. Find the envelopes of each of the following families of curves:

(a) $y = 4kx - k^4$. *Ans.* $y^3 = 27x^4$.

(b) $y^2 = kx - k^2$. *Ans.* $y = \pm \frac{1}{2}x$.

(c) $y = kx \pm \sqrt{1 + k^2}$. *Ans.* $x^2 + y^2 = 1$.

(d) $y^2 = k^2x - 2k$. *Ans.* $xy^2 = -1$.

(e) $(x - k)^2 + y^2 = 2k$. *Ans.* $y^2 = 2x + 1$.

(f) $4x^2 + (y - k)^2 = 1 - k^2$. *Ans.* $y^2 + 8x^2 = 2$.

(g) $x \cos\theta + y \sin\theta = 10$. *Ans.* $x^2 + y^2 = 100$.

3. Show that the normal to the curve $y = x^3$ at any point (k, k^3) on it, is $y - k^3 = -(x - k)/3k^2$. Hence show that the evolute of $y = x^3$ is given by the parameter equations $x = (k - 9k^5)/2$, $y = (15k^4 + 1)/(6k)$.

4. Taking the equations of an ellipse of semiaxes a and b in the form $x = a\cos\theta$, $y = b\sin\theta$, show that the equation of the normal at any point is $by = ax\tan\theta + (b^2 - a^2)\sin\theta$. Hence show that the evolute of the ellipse is given by the parameter equations $ax = (a^2 - b^2)\cos^3\theta$, $by = (b^2 - a^2)\sin^3\theta$.

5. Find the evolute of the curve $y^2 = x^3$.

6. Find the evolute of the curve $y = e^x$.

7. Show that the envelope of a family of circles through the origin with their centers on the parabola $y^2 = 2x$ is $y^2(x + 1) + x^3 = 0$.

8. Show that the envelope of the family of straight lines $ax + by = 1$ where $a + b = ab$, is the parabola $x^{1/2} + y^{1/2} = 1$.

9. Show that the envelope of the family of parabolas $y = x\tan\alpha - mx^2\sec^2\alpha$ is $y = 1/(4m) - mx^2$.

[NOTE. If $m = g/(2v_0^2)$, the given equation represents the path of a projectile fired from the origin with initial speed v_0 at an angle of elevation α.]

10. Find the evolute of the curve $y = (e^x + e^{-x})/2$.

154. Properties of Evolutes. In general, taking the equations (8)′ of § 153, the equation of the evolute of a given curve $y=f(x)$ may be written:

$$(1) \qquad x - x_k = -\frac{m_k(1+m_k^2)}{b_k}, \quad y - y_k = \frac{1+m_k^2}{b_k},$$

where $x_k=k$, $y_k=f(k)$, $m_k=f'(k)$ (the slope at $x=k$), $b_k=f''(k)$ (the flexion at $x=k$). The point (x_k, y_k) lies on the given curve $y=f(x)$; the point (x, y) lies on the evolute; the normal at (x_k, y_k) to the given curve is tangent to the evolute at (x, y). Hence the distance D, measured along the normal, from the given curve to the point of tangency on the evolute, is given by the equation:

$$D^2 = (x-x_k)^2 + (y-y_k)^2 = \frac{m_k^2(1+m_k^2)^2}{b_k^2} + \frac{(1+m_k^2)^2}{b_k^2}$$
$$= \frac{(1+m_k^2)^3}{b_k^2};$$

it follows that D is precisely the radius of curvature (see § 97, and Ex. 4, p. 172):

$$D = \frac{(1+m_k^2)^{3/2}}{b_k} = R.$$

Hence the *radius of curvature* of a curve is shown graphically when the evolute is drawn: in Fig. 68, p. 300, for example, the radii of curvature at A, B, C, D, O are the lengths AA', BB', CC', DD', OO', respectively. Notice particularly that the *change* in the radius of curvature can be followed by the eye very clearly by means of the evolute, as the point on the given curve moves.*

155. Center of Curvature. The point at which the normal to the given curve is tangent to the evolute is at a distance R from the given curve (along the normal); this point is called

* This is of importance in laying out railroad curves, etc., where the change in the radius is of great moment; in particular the minimum value of the radius is often important.

the **center of curvature**; its coördinates are precisely the x and y of equations (1), § 154. (See Ex. 2, p. 171.)

The angle β which the normal makes with the x-axis is shown in Fig. 68, p. 300; from the right triangle CAC' we have

$$\angle ACC' = \beta,\ CC' = R,\ CA = x - x_k,\ AC' = y - y_k,$$

and therefore

(1) $$x - x_k = R \cos\beta,\ y - y_k = R\sin\beta;$$

moreover

(2) $$\frac{y-y_k}{x-x_k} = \tan\beta = -\frac{1}{m_k} = -\frac{1}{dy_k/dx_k} = \frac{dy}{dx},$$

since $\tan\beta$ is the slope of the normal $(= -1/m_k)$ of the given curve, and also the slope of the tangent of the evolute.

The circle whose radius is R (the radius of curvature) and whose center is the point (x, y) at which the normal is tangent to the evolute, is called the **circle of curvature**; its equation is $(X - x)^2 + (Y - y)^2 = R^2$, where (x, y) is the fixed point on the evolute and (X, Y) is the variable point on the circle of curvature.

156. Rate of Change of R. The rate at which R changes, which was mentioned in § 154, can be obtained as follows. Since

$$R^2 = (x - x_k)^2 + (y - y_k)^2,$$

we have

(2) $$R\,dR = (x - x_k)(dx - dx_k) + (y - y_k)(dy - dy_k),$$

or

(3) $$dR = \frac{(x-x_k)\,dx + (y-y_k)\,dy}{\sqrt{(x-x_k)^2 + (y-y_k)^2}},$$

since $$(x - x_k)\,dx_k + (y - y_k)\,dy_k = 0, \qquad \text{by (2), § 155.}$$

But since $$(y - y_k)/(x - x_k) = dy/dx, \qquad \text{by (2), § 155,}$$

(4) $$dR = \frac{dx + \dfrac{y-y_k}{x-x_k}dy}{\sqrt{1 + \left(\dfrac{y-y_k}{x-x_k}\right)^2}} = \frac{dx + \dfrac{dy}{dx}dy}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}} = \sqrt{dx^2 + dy^2},$$

and since $\sqrt{dx^2 + dy^2} = ds$, where s is the length of arc of the evolute,

(5) $$dR = ds, \quad \text{or} \int_{R=R_1}^{R=R_2} dR = \int_{s=s_1}^{s=s_2} ds, \quad \text{or } R_2 - R_1 = s_2 - s_1;$$

that is: the rate of growth of the radius of curvature is equal to the rate of growth of the arc of the evolute; and the difference between two radii of curvature is the same as the length of the arc of the evolute which separates them.

This fact gives rise to an interesting method of drawing the original curve (the involute) from the evolute: Imagine a string wound along the convex portion of the evolute, fastened at some point (say D', Fig. 68, p. 300) and then stretched taut. If a pencil is inserted at any point (say C, Fig. 68) in the string, the pencil will traverse the involute as the string, still held taut, is unwound from the evolute.

157. Illustrative Examples.

Example 1. The evolute of the curve $y = x^2$ was found in § 153 to be

$$x = -4k^3, \quad y = 3k^2 + 1/2.$$

The radius of curvature of the given curve at the point ($x_k = k$, $y_k = k^2$) is therefore

$$\begin{aligned} R &= \sqrt{(x-k)^2 + (y-k^2)^2} = \sqrt{(-4k^3 - k)^2 + (2k^2 + 1/2)^2} \\ &= \tfrac{1}{2}(4k^2 + 1)^{3/2}. \end{aligned}$$

The rate of change of R with respect to k is

$$\frac{dR}{dk} = 6k(4k^2 + 1)^{1/2};$$

and R is a maximum or a minimum only where this rate is zero, *i.e.* where $k = 0$. Since dR/dk is negative when $k < 0$, and positive when $k > 0$, it follows that R is a minimum when $k = 0$, *i.e.* at the point O in Fig. 68, p. 300. The value of R at this point is $R_0 = 1/2$; this is also evident in the figure.

Example 2. To find the evolute and radius of curvature of the cycloid;

(1) $$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$$

The slope m_k at a point (x_k, y_k) where $t = k$ is

$$m_k = \frac{dy_k}{dx_k} = \frac{\dfrac{dy_k}{dt}}{\dfrac{dx_k}{dt}} = \frac{a \sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t} = \operatorname{ctn} \frac{t}{2}.$$

and the second derivative $b_k = d^2y_k/dx^2_k$ is

$$b_k = \frac{dm_k}{dx_k} = \frac{\frac{dm_k}{dt}}{\frac{dx_k}{dt}} = \frac{\frac{\cos t\,(1-\cos t) - \sin^2 t}{(1-\cos t)^2}}{a(1-\cos t)} = \frac{-1}{a\,(1-\cos t)^2} = \frac{-1}{4\,a\sin^4(t/2)}.$$

It follows that the evolute is given by the equations:

$$x = x_k - \frac{m_k(1+m_k^2)}{b_k} = a(t-\sin t) + 2\,a\sin t = a(t+\sin t),$$

$$y = y_k + \frac{1+m_k^2}{b_k} = a(1-\cos t) - 2\,a(1-\cos t) = -\,a(1-\cos t)$$

which is another cycloid of the same shape and size, with its vertices at the points $y = -\,2\,a$, $x = \pi a$, $3\,\pi a$, etc., as shown in Fig. 69.

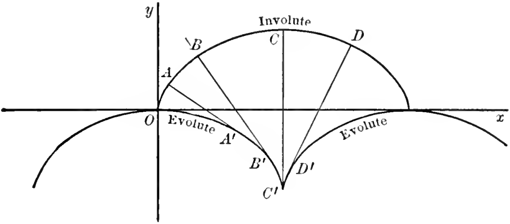


FIG. 69

The radius of curvature is given by the equation

$$R^2 = (x-x_k)^2 + (y-y_k)^2 = (2\,a\sin t)^2 + (-\,2\,a(1-\cos t))^2$$
$$= 8\,a^2(1-\cos t),$$

whence $$R = 4\,a\sqrt{\frac{1-\cos t}{2}} = 4\,a\sqrt{\sin^2\left(\frac{t}{2}\right)} = 4\,a\sin\left(\frac{t}{2}\right).$$

The value of R at O is given by $t = 0$: $R_0 = 0$; the value of R at C is given by $t = \pi$: $R_C = 4\,a$. Since the arc OC' of the evolute is equal to the difference of these values of R, we have arc $OC' = 4\,a$; hence the length of a whole arch of the cycloid is $2 \cdot OC' = 8\,a$ (Ex. 16, p. 155).

EXERCISES LXIV. — PROPERTIES OF EVOLUTES

1. Find the general equation of the circle of curvature for the curve $y = x^3$. Draw it for the points (1, 1), (2, 8), (1/2, 1/8).

2. Find the radius of curvature, the circle of curvature, and the evolute, for any point on the curve $x = 4 \cos \theta$, $y = \sin \theta$.

3. Find the radius of curvature, the circle of curvature, and the evolute (in parameter form) for each of the following curves at any point:

(a) $\begin{cases} x = \sin \theta, \\ y = 2 \cos \theta. \end{cases}$ (d) $\begin{cases} x = t, \\ y = \cos t. \end{cases}$

(b) $\begin{cases} x = \sec \theta, \\ y = \tan \theta. \end{cases}$ (e) $\begin{cases} x = \cos t + t \sin t, \\ y = \sin t - t \cos t. \end{cases}$

(c) $\begin{cases} x = 1/t, \\ y = 2t. \end{cases}$ (f) $\begin{cases} x = 2 + 3t, \\ y = t^2 - 4. \end{cases}$

4. Find the minimum value of the radius of curvature for the curve $y = x^3$. *Ans.* $R = (3/5)\sqrt[4]{4/5}$.

5. Show that the length of one quarter of the evolute of an ellipse is $(a^3 - b^3)/ab$.

6. Show that the curvature of a curve at any point of inflexion is zero.

7. The curvature $K = 1/R$ is obtained by multiplying the flexion b by the corrective factor $(1 + m^2)^{-3/2}$. Show that this corrective factor is equal to $\cos^3 \alpha$, where α is the angle between the given curve and the x axis. (See § 97, p. 169.)

8. Show directly from the definition of § 97 that $K = d\alpha/ds = b \cos^3 \alpha$.

[Hint. $m = \tan \alpha$, hence $dm = \sec^2 \alpha \; d\alpha$; but $dx/ds = \cos \alpha$ and $dm/dx = b$.]

9. Find the equation of the evolute, and its length of arc, for the tractrix, $y = a \sin \theta$, $x = a \log \cot (\theta/2) - a \cos \theta$.

10. The evolute of the equiangular spiral $\rho = ae^{k\theta}$ is an equal curve turned through an angle $\pi/2 + k \log k$. Determine the length of arc.

11. When a thread is suspended from one cusp of a horizontal inverted cycloid, the thread carrying a weight at the free end and having a length equal to twice the height of arch of the curve, show that if the weight be made to oscillate so that the thread winds up on the curve, it will describe an equal cycloid.

12. Construct the evolutes of the curves,

$$(a)\ y = \sin x, \quad (b)\ y = \tan x, \quad (c)\ y = a \cosh (x/a).$$

13. The lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ may be written:

$$x = a \cos t \sqrt{\cos 2t}\,; \quad y = a \sin t \sqrt{\cos 2t}.$$

Show that the evolute is $(x^{2/3}+y^{2/3})^2\,(x^{2/3} - y^{2/3}) = 4\,a^2/9.$

158. Singular Points. The tangent to a curve whose equation is given in explicit form $y = f(x)$, where $f(x)$ is single-valued, can neither be vertical nor fail to exist if $dy/dx = f'(x)$ exists. If the equation of the curve is given in the implicit form,

$$(1) \qquad f(x, y) = 0,$$

the slope of the tangent, $m = dy/dx$, is given by the equation (Example 3, § 151, p. 294):

$$(2) \qquad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0,$$

which can be solved for dy/dx as in § 151, *unless* $\partial f/\partial y = 0$. At a point S at which $\partial f/\partial y = 0$, if $\partial f/\partial x \neq 0$, the tangent is vertical; if both $\partial f/\partial y$ and $\partial f/\partial x$ are zero at S, the equation (2) is practically useless, and the tangent may be indeterminate.

For this reason, a point S on the curve (1) for which $\partial f/\partial x$ and $\partial f/\partial y$ both vanish is called a **singular point**; such points may be found by solving the equations

$$(3) \qquad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0,$$

as simultaneous equations for x and y. The points thus found may not lie on the curve (1); if not, they should be discarded.

Notice, however, that any pair of solutions (a, b) of (3) are the coördinates of a singular point of the contour line of the surface $z = f(x, y)$ cut out by the plane $z = c$, where $f(a, b) = c$.

Usually a due amount of care in plotting the curve near the singular point will indicate its nature. A detailed discussion is

given in advanced texts on Calculus.* The points of (3) for which $\partial f/\partial y$ alone vanishes should also be inspected carefully

159. Illustrative Examples.

Example 1. Examine the curve $x^3+y^3-3xy=0$ for singular points.

In this example $f(x, y) = x^3+y^3-3xy=0$. In § 27, p. 45 we found dy/dx for this curve by a process which amounts to the same thing as writing

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=(3x^2-3y)+(3y^2-3x)\frac{dy}{dx}=0.$$

This equation can be solved for dy/dx [or for dx/dy] *unless*

$$x^2-y=0, \quad y^2-x=0.$$

These equations have the two pairs of solutions $(x=0,\ y=0)$ and $(x=1,\ y=1)$. The point $(0,\ 0)$ lies on the curve, and is therefore a singular point. A careful figure, drawn as in Ex. 12, p. 63, shows that the curve crosses itself at this point, and has no single tangent; such a point is called a **double point**. (See *Tables*, III, I_5.)

The point $(1,\ 1)$ does not lie on the given curve since $f(1,\ 1)=1+1-3=-1$. But it does lie on the contour line of the surface $z=x^3+y^3-3xy$ cut out by the plane $z=-1$. A careful graph of this contour line $x^3+y^3-3xy=-1$ reveals the fact that there is no other point on the curve near $(1,\ 1)$, although there is another portion of the curve some distance away; such a point is called an **isolated point**. The plotting of the figures is facilitated by first rotating the xy-axes through $45°$.

Example 2. Examine the curve $y^3=x^2$ for singular points.

Here $f(x, y)=y^3-x^2=0$, and we write:

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=-2x+3y^2\frac{dy}{dx}=0,$$

an equation which determines dy/dx [or dx/dy] except when $x=y=0$. This point $(0,\ 0)$ lies on the given curve; hence it is a singular point. Careful plotting (see *Tables*, III, A) near the point indicates that the curve has a sharp **corner** at this point. At any other point $dy/dx=2x/(3y^2)=2/(3x^{1/3})$. As x approaches zero from either side, this quantity becomes infinite. Hence the tangent approaches a vertical position as x approaches zero from either side. A corner is called a **cusp** if the two branches of the curve which meet there have, as here, a common tangent line.

* See, *e.g.*, Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 110.

Example 3. Examine the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t)$$

for singular points.

The value of dy/dx [or dx/dy] is given, as above, by the equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1-\cos t)} = \frac{\sin t}{1 - \cos t},$$

unless $\sin t = 0$ and $1 - \cos t = 0$; these equations are both satisfied when $t = 0$, $\pm 2\pi$, etc., (not at $t = \pi$). Hence the points where $t = 0$, [*i.e.* $(x = 0, y = 0)$], $t = 2\pi$ [*i.e.* $(x = 2\pi a, y = 0)$], etc., are singular points. It results from the rules for indeterminate forms (§ 136, p. 263) that

$$\lim_{t \doteq 0} \frac{dx}{dy} = \lim_{t \doteq 0} \frac{1 - \cos t}{\sin t} = \lim_{t \doteq 0} \frac{\sin t}{\cos t} = 0;$$

hence dx/dy approaches zero as t approaches zero [*i.e.* as (x, y) approaches $(0, 0)$]; therefore the tangent becomes more and more nearly vertical as we approach the singular point $(0, 0)$ from either side; the singular points of a cycloid are therefore cusps.

When the equations of a curve are given in parameter form, the singular points can be located as in this example, by finding the common solutions of the equations $dx/dt = 0$, $dy/dt = 0$; and it is usually possible to determine, by the rules for indeterminate forms, what happens to the tangent as that point is approached.

160. Asymptotes. The search for the asymptotes of a curve is often facilitated by our knowledge of the Calculus.

Vertical or horizontal asymptotes are usually best found by the purely algebraic methods of analytic geometry. Thus if $f(x)$ is a fraction, the curve $y = f(x)$ has a vertical asymptote $x = k$ if a factor of the denominator vanishes when $x = k$. (See, however, § 139, p. 268). If $f(x)$ has a factor $\tan x$ or $\log x$ or $\sec x$, $\cdots$, $y = f(x)$ may have a vertical asymptote at any point where that factor becomes infinite. Useful rules for horizontal asymptotes result by interchange of x and y.

If the asymptote is neither horizontal nor vertical, these elementary means are insufficient. If the tangent to a given curve:

(1) $$y - y_P = m_P(x - x_P),$$

at the point P, (x_P, y_P), approaches a fixed limiting position

$$y = ax + b \tag{2}$$

as the distance OP from the origin to P becomes infinite, the line (2) is called an **asymptote**. This will be true if and only if

$$\lim_{x_P \doteq \infty} m_P = a, \text{ and } \lim_{x_P \doteq \infty} (y_P - m_P x_P) = b,$$

where a and b are constants. The value of m_P can be computed by any of our usual methods and then $\lim m_P$ can be found if it exists. In this work it is useful to notice that *the ratio* $[y/x]_P$ *also approaches* a if there is actually an asymptote (2) which is not vertical.

Example 1. Examine curve $x^3 + y^3 - 3\,xy = 0$ for asymptotes. The method used in Ex. 1, p. 310, gives

$$m_P = \frac{dy}{dx}\Big]_P = \frac{x^2 - y}{x - y^2}\Big]_P;$$

hence

$$\lim_{x_P \doteq \infty} m_P = \lim_{x_P \doteq \infty} \frac{1 - \left(\dfrac{y}{x}\right)\dfrac{1}{x}}{\dfrac{1}{x} - \left(\dfrac{y}{x}\right)^2}\Bigg]_P = \lim_{x_P \doteq \infty} \frac{-1}{\left(\dfrac{y}{x}\right)^2}\Bigg]_P = -\frac{1}{a^2}$$

since $1/x_P$ approaches zero, and $[y/x]_P$ approaches a if a exists. Since $\lim m_P = a$, we have $a = -1/a^2$, if a exists, whence $a = -1$.

The equation of the given curve may be written in the form

$$\left(\frac{y}{x}\right)^3 = -1 + 3\left(\frac{y}{x}\right)\frac{1}{x},$$

whence it is evident that y/x does approach -1 as x becomes infinite. Finally the expression $y_P - m_P x_P$ becomes

$$y_P - m_P x_P = \frac{2\,xy - x^3 - y^3}{x - y^2}\Big]_P = \frac{-xy}{x - y^2}\Big]_P = \frac{y/x}{(y/x)^2 - 1/x}\Big]_P;$$

hence

$$\lim_{x_P \doteq \infty} (y_P - m_P x_P) = \lim_{x_P \doteq \infty} \frac{\dfrac{y}{x}}{\left(\dfrac{y}{x}\right)^2 - \dfrac{1}{x}}\Bigg]_P = -1,$$

since $\lim\,[y/x]_P = -1$ and $\lim\,[1/x]_P = 0$. The values of a and b are therefore $a = -1$, $b = -1$, and the line $y = -x - 1$ is an asymptote.

The knowledge of this fact assists materially in drawing an accurate figure.

In general, an equation of the form $f(x, y) = 0$ gives

$$m = -(\partial f/\partial x) \div (\partial f/\partial y).$$

If $f(x, y)$ is algebraic, the value of m can be arranged as above in powers of (y/x) and $(1/x)$ [or of (x/y) and $(1/y)$]; and the equation $f(x, y) = 0$ can also be written in terms of (y/x) and $(1/x)$. The work in any case is similar to that of the preceding example.

161. Curve Tracing. In order to draw a curve whose equation is given, it is often desirable to find whether there are any asymptotes or any singular points before an attempt is made to draw the curve. It is also useful to know the positions of any maxima and minima (§§ 37, 47, 135) and of any points of inflexion (§ 46, p. 75). The actual construction of a few tangents is often useful, particularly at points of inflexion.

Elementary methods should not be abandoned ruthlessly. Building up a graph by adding, multiplying, or dividing the ordinates of two simpler curves; moving a curve vertically or horizontally; increase or decrease of scale on one axis at a time; plotting from equations in parameter form; in some rare instances, rotation of axes; in all cases, *inspection of the given equation for possible simplifications:* these elementary methods are even more fundamental and vital than the newer ideas explained above.

EXERCISES LXV. — SINGULAR POINTS, ASYMPTOTES, CURVE TRACING

1. Find the asymptotes A, and the singular points S for each of the following curves; then trace each curve. Use elementary methods whenever possible, and use the points of inflexion and the extremes, if any exist. In every case, try to build up the curve from simpler ones; in most of these exercises, this can be done.

(*a*) $y=\dfrac{1}{a-x}$; $A: x=a$; no S.

(*b*) $y^2=\dfrac{1}{x-a}$; $A: x=a$; no S.

(*c*) $y^2=x^2-x^4$; no A; $S:(0,0)$, double point.

(*d*) $y^3=2x^2-x^3$; $A: y=-x+2/3$; $S:(0,0)$, cusp.

(*e*) $x=y(x-a)^2$; $A: x=a,\ y=0$; no S.

(*f*) $y^2(2a-x)=x^3$; $A: x=2a$; S: $(0,0)$, cusp.

(*g*) $x^2y=4a^2(2a-y)$; $A: y=0$; no S.

(*h*) $y^3=9x^2+x^3$; $A: y=x+3$; $S:(0,0)$, cusp.

(*i*) $y^2(x^2+1)=x^2(x^2-1)$; $A: y=\pm x$; $S:(0,0)$, isolated.

(*j*) $y^2(x-2)=x^3-1$; $A: x=2,\ y=\pm(x+1)$; no S.

(*k*) $y=e^x$; $A: y=0$; no S.

(*l*) $y=(e^x+e^{-x})/2$; no A; no S.

(*m*) $y=e^{-x}$; $A: y=0$; no S.

(*n*) $y=\sec x$; $A: y=n\pi/2$, n any odd integer; no S.

2. Show that the curve $y=2/(e^x+e^{-x})=\operatorname{sech} x$ is asymptotic to the x-axis, by building up its graph from that of 1 (*l*).

3. Show that each of the curves

$$y=xe^{-x},\quad y=x^2e^{-x},\quad y=x^3e^{-x},\ \cdots,\ y=x^ne^{-x},$$

is asymptotic to the x-axis (See Exs. 3, 5, p. 271).

4. Show that the curve 1 (*a*) has no area, in the sense of § 111, between $x=a$ and $x=a+1$, nor from $x=a+1$ to $x=\infty$.

5. Show that the curve 1 (*b*) has an area between $x=a$ and $x=a+1$, but not from $x=a+1$ to $x=\infty$.

6. Show that the curve of Ex. 1 (*e*) has no area between $x=a$ and $x=a+1$, and has no area from $x=a+1$ to $x=\infty$.

7. Show that the curve $y=x\log x$ ends abruptly at the origin, by building up its graph. [See § 140, p. 269.]

8. Show that the curve $y=e^{-x}\sin x$ is asymptotic to the x-axis.

9. Show that $y=\sin(1/x)$ has an infinite number of maxima and minima near the origin; and that it is asymptotic to the x-axis.

10. Build up the graph of $y=x\sin(1/x)$ from Ex. 9.

11. Show that the curves $y=(e^x+e^{-x})/2=\cosh x$ and $y=(e^x-e^{-x})/2=\sinh x$ are asymptotic to each other, and to the curve $y=e^x/2$ as x becomes infinite. See *Tables*, III, E.

12. Build up the graph of $y=e^{-1/x^2}$; show that it is asymptotic to the line $y=1$.

PART III. GEOMETRY OF SPACE EXTREMES

162. Résumé of Formulas of Solid Analytic Geometry.

(*a*) **Distance between two Points** (x_1, y_1, z_1) **and** (x_2, y_2, z_2):

(1) $$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

(*b*) **Distance from Origin to** (x, y, z):

$$r=\sqrt{x^2+y^2+z^2}.$$

(*c*) **Direction Cosines.** If α, β, γ denote the angles that a given line makes with the positive directions of the x, y, z axes respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the **direction cosines** of the given lines; and we always have

(2) $$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

If the direction cosines are *proportional to three numbers* a, b, c, their actual values are

(3) $$\cos\alpha=\frac{a}{\sqrt{a^2+b^2+c^2}},\quad \cos\beta=\frac{b}{\sqrt{a^2+b^2+c^2}},$$
$$\cos\gamma=\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

If we indicate the direction cosines by single letters, say

(4) $$l=\cos\alpha,\quad m=\cos\beta,\quad n=\cos\gamma,$$

we speak of the direction (l, m, n).

(*d*) **Angle between Two Directions.** The angle θ between the directions (l, m, n) and (l', m', n') is given by

(5) $$\cos\theta=ll'+mm'+nn'.$$

The directions are *parallel*, if

(6) $$ll'+mm'+nn'=1.$$

They are *perpendicular*, if

(7) $$ll'+mm'+nn'=0.$$

(*e*) **The Plane.** If p is the length of the perpendicular from the origin upon a plane and (l, m, n) is its direction, the plane is denoted by $(l, m, n;\ p)$, and its equation is

$$(8)\qquad lx + my + nz = p, \quad \text{or} \quad x \cos\alpha + y \cos\beta + z \cos\gamma = p.$$

Since the distance d from the plane $(l, m, n;\ p)$ to the point (x_1, y_1, z_1) is

$$(9)\qquad d = lx_1 + my_1 + nz_1 - p;$$

the form (8) is called the *distance* form, or the *normal* form, of the equation of a plane.

If the axial *intercepts* of a plane are a, b, c, its equation is

$$(10)\qquad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

The plane through the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is *

$$(11)\qquad \begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix} = 0.$$

The general equation of the plane is the *general equation of the first degree*, namely:

$$(12)\qquad Ax + By + Cz + D = 0.$$

If the direction of the normal to the plane from the origin is (l, m, n), and its distance from the origin is p, we have

$$(13)\qquad \begin{aligned} l &= \frac{A}{\sqrt{A^2 + B^2 + C^2}}, & m &= \frac{B}{\sqrt{A^2 + B^2 + C^2}}, \\ n &= \frac{C}{\sqrt{A^2 + B^2 + C^2}}, & p &= \frac{-D}{\sqrt{A^2 + B^2 + C^2}}. \end{aligned}$$

The angle between two planes is the angle between their

* The definition of a determinant is given in the *Tables*, II, C, 5.

normals; it is given by

$$(14)\qquad \cos\theta = \frac{AA' + BB' + CC'}{\sqrt{(A^2 + B^2 + C^2)(A'^2 + B'^2 + C'^2)}}.$$

The planes are *parallel*, if

$$A/A' = B/B' = C/C';$$

they are *perpendicular*, if

$$AA' + BB' + CC' = 0.$$

The *distance* from the plane $Ax + By + Cz + D = 0$ to the point (x_1, y_1, z_1) is

$$(15)\qquad d = \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}.$$

(*f*) **The Straight Line.** In general, a straight line is represented by the intersection of two planes:

$$(16)\qquad Ax + By + Cz + D = 0, \quad A'x + B'y + C'z + D' = 0.$$

The direction cosines of the line are given by the proportion

$$(17)\qquad l : m : n = \begin{vmatrix} B & C \\ B' & C' \end{vmatrix} : \begin{vmatrix} C & A \\ C' & A' \end{vmatrix} : \begin{vmatrix} A & B \\ A' & B' \end{vmatrix},$$

together with the principle (3).

The equations of the straight line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$(18)\qquad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

The line through the point (x_1, y_1, z_1) in the direction (l, m, n), is

$$(19)\qquad \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

(g) **Quadric Surfaces. Equations of the Second Degree.**

Spheres, center (a, b, c), radius r:

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2. \tag{20}$$

Cones, vertices at origin:

$$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=0. \tag{21}$$

(*Imaginary,* if all signs are alike; *otherwise real,* and sections parallel to *one* of the reference planes elliptic.)

Ellipsoids and **hyperboloids,** centers at origin (*Tables,* III, N):

$$\pm\frac{x^2}{a^2}\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=1. \tag{22}$$

All signs on the left $+$, *ellipsoid.*

One sign on the left $-$, *hyperboloid of one sheet.*

Two signs on the left $-$, *hyperboloid of two sheets.*

Three signs on the left $-$, *imaginary.*

Paraboloids, vertices at the origin (*Tables,* III, $N_{4,5}$):

$$\pm\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=cz. \tag{23}$$

Like signs, *elliptic* paraboloid; unlike, *hyperbolic* paraboloid.

163. Loci of One or More Equations in Three Variables.

A single equation in three variables,

$$F(x, y, z)=0, \tag{1}$$

represents, in general, **a curved surface in space.** If z is given a series of constant values $a_1, a_2, a_3, \cdots$ successively, the coördinates x, y will satisfy the equations of the curves

$$F(x, y, a_1)=0, \quad F(x, y, a_2)=0, \quad F(z, y, a_3)=0, \quad \cdots \tag{2}$$

in which the planes

$$z=a_1, \quad z=a_2, \quad z=a_3, \quad \cdots \tag{3}$$

cut the surface. These curves are, in fact, **contour lines on the surface**, and the totality of them, for all possible values of z, makes up the surface.

Two independent simultaneous equations:

(4) $$f(x, y, z) = 0, \quad \phi(x, y, z) = 0,$$

are satisfied, in general, by the intersection of two surfaces, and therefore represent **a curve in space.**

Three independent simultaneous equations,

(5) $$f(x, y, z) = 0, \quad \phi(x, y, z) = 0, \quad \psi(x, y, z) = 0,$$

are true, in general, only at certain **isolated points**; those, namely, in which the curve represented by two of the equations cuts the surface represented by the third.

A single equation from which one of the coördinates is missing is a **cylinder** *with axis parallel to the axis of the missing coördinate.* Thus

(6) $$f(x, y) = 0,$$

interpreted in space, is a cylinder parallel to the z-axis. Its trace on the xy-plane is the plane curve,

(7) $$f(x, y) = 0, z = 0.$$

EXERCISES LXVI. — RÉSUMÉ OF SOLID GEOMETRY

1. Find a straight line through each of the following pairs of points; find its direction cosines.

(*a*) (0, 1, 0) and (2, 3, 5).
(*b*) (– 1, 2, – 3) and (2, – 1, 0).
(*c*) (4, 1, – 5) and (2, 1, – 3).
(*d*) (5, 3, 7) and (5, – 2, 7).

2. Find the direction cosines of each of the following planes:

(*a*) $2x - 3y + 4z = 5$.
(*b*) $x + y + z = 0$.
(*c*) $y - 3z = 2$.
(*d*) $z = 2x - y + 4$.

3. Find the equations of a line formed by the intersection of the planes 2 (*a*) and 2 (*b*), in the form (19), and find its direction cosines.

4. Proceed as in Ex. 2 for each of the combinations formed by two of the planes mentioned in Ex. 2.

5. Reduce each of the equations in Ex. 2 to normal form; find the distance from each of these planes to the origin.

6. Find the equation of a plane through the origin which

(a) also passes through the two points of Ex. 1 (a);

or (b) is parallel to the plane 2 (a);

or (c) is perpendicular to each of the planes 2 (a) and 2 (c).

7. Find the angle between each pair of planes in Ex. 2.

8. Find the angle between the *direction* specified by Ex. 1 (a) and that specified by Ex. 1 (b); between the directions specified by each pair of lines mentioned in Ex. 1.

9. Find the center and the radius of each of the following spheres:

(a) $x^2 + y^2 + z^2 + 2x - 4y + 6z = 2$.

(b) $x^2 + y^2 + z^2 + 12x - y - 4z + 40 = 0$.

10. Find the equation of a sphere

(a) whose center is $(2, -1, 4)$ and whose radius is 3;

(b) one of whose diameters joins $(2, 4, -1)$ and $(3, 1, 6)$;

(c) whose center is $(1, 0, 5)$ and which passes through $(3, 1, -2)$.

11. Reduce to standard form and identify each of the following surfaces:

(a) $x^2 + 4y^2 + z^2 - 6x + 2z = 6$.
(d) $9x^2 - y^2 + 4z^2 + 6x + 10y = -25$.

(b) $x^2 - 4y^2 - 6x + 2z = 6$.
(e) $4x^2 - y^2 - 4x + 6y = 15$.

(c) $9x^2 - y^2 + 4z^2 + 6x + 10y = 10$.
(f) $z^2 + 9x^2 - 2z + 4y = 0$.

12. Represent each of the following equations or groups of equations geometrically in space of 3 dimensions; find the trace, if any exists, on each coördinate plane, and on each of a series of parallel planes:

(a) $z = xy$. (b) $x^2 = y^2 + z^2$. (c) $x^2 + y^2 + z^2 = 1$. (d) $y = \sin x$.

(e) $xyz = 1$. (f) $x^2 + y^2 = \sin z$. (g) $x + y = e^z$. (h) $z = e^{x+y}$.

(i) $x^2 + y^2 + z^2 = 4,\ x + y = 0$. ($j$) $z^2 = x^2 + y^2,\ z = 1 - x$.

(k) $x = \cos z,\ y = \sin x$. (l) $x + y = e^z,\ y = 2x$.

(m) $y = z^2,\ x = y^2$. (n) $x^2 = z - y,\ x + y = 0,\ z + y = 0$.

(o) $x^2 - y^2 = 4z,\ x - y = 4,\ x + y = 7$.

(p) $x + y = z,\ y + z = x,\ z + x = y$.

164. Tangent Plane to a Surface. Let P_0 be the point (x_0, y_0, z_0) on the surface $z = f(x, y)$. Let P_0T_1 be the tangent line at P_0 to the curve cut from the surface by the plane $y = y_0$ and P_0T_2 the tangent line to the curve cut from the surface by the plane $x = x_0$. The plane containing these two lines is the tangent plane to the surface at P_0.

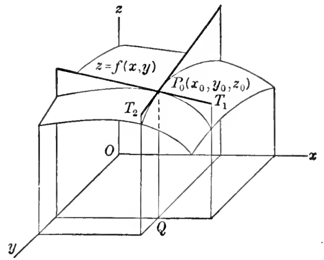


FIG. 70

Since this plane goes through P_0, its equation can be thrown into the form

$$(1)\qquad z - z_0 = A(x - x_0) + B(y - y_0).$$

If we set $y = y_0$ we find the equation of P_0T_1 in the form:

$$(2)\qquad z - z_0 = A(x - x_0).$$

But, from § 33, p. 58, the equation of P_0T_1 may be written in the form:

$$(3)\qquad z - z_0 = \left.\frac{\partial f}{\partial x}\right]_0 (x - x_0).$$

Hence

$$(4)\qquad A = \left.\frac{\partial f}{\partial x}\right]_0; \quad \text{likewise } B = \left.\frac{\partial f}{\partial y}\right]_0.$$

Thus the **equation of the tangent plane** is

$$(5)\qquad \boldsymbol{z - z_0 = \left.\frac{\partial f}{\partial x}\right]_0 (x - x_0) + \left.\frac{\partial f}{\partial y}\right]_0 (y - y_0)};$$

or, what is the same thing,

$$(6)\qquad z - z_0 = \left.\frac{\partial z}{\partial x}\right]_0 (x - x_0) + \left.\frac{\partial z}{\partial y}\right]_0 (y - y_0).$$

It is important to notice the great similarity between this equation and the equation

$$(7)\qquad dz = \left.\frac{\partial z}{\partial x}\right]_0 dx + \left.\frac{\partial z}{\partial x}\right]_0 dy,$$

of § 148. In fact (7) expresses the fact that if dx, dy are measured parallel to the x and y axes from the point of tangency (x_0, y_0, z_0), dz represents the height of the tangent plane above (x_0, y_0, z_0). Equation (7) furnishes a good means of remembering (6).

165. Extremes on a Surface. If a function $z=f(x, y)$ is represented geometrically by a surface, it is evident that the extreme values of z are represented by the points on the surface which are the *highest*, or the *lowest*, points in their neighborhood:

(1) $f(x_0, y_0) > f(x_0+h, y_0+k)$, if $f(x_0, y_0)$ is a *maximum*,

(2) $f(x_0, y_0) < f(x_0+h, y_0+k)$, if $f(x_0, y_0)$ is a *minimum*,

for all values of h and k for which h^2+k^2 is not zero and is not too large.

It is evident directly from the geometry of the figure that *the tangent plane at such a point is horizontal.*

This results also, however, from the fact that the section of the surface by the plane $x = x_0$ must have an extreme at (x_0, y_0); hence $[\partial f/\partial y]_0$, which is the slope of this section at (x_0, y_0), must be zero; likewise $[\partial f/\partial x]_0$, the slope of the section through (x_0, y_0) by the plane $y = y_0$, must be zero. Hence equation (5), § 164, reduces to $z - z_0 = 0$, which is a horizontal plane.

A point at which the tangent plane is horizontal is called a **critical point** on the surface. The following cases may present themselves.

(1) *The surface may cut through its tangent plane; then there is* **no extreme** *at* (x_0, y_0).

This is what happens at a point on a surface of the saddleback type shown by a hyperbolic paraboloid at the origin; a homelier example is the depression between the knuckles of a clenched fist.

(2) *The surface may just touch its tangent plane along a whole line, but not pierce through; then there is what is often called a* **weak extreme** *at* (x_0, y_0); that is, $z=f(x, y)$ has the same value along a whole line that it has at (x_0, y_0), but otherwise $f(x, y)$ is less than [or greater than] $f(x_0, y_0)$.

This is what happens on the top of a surface which has a rim, such as the upper edge of a water glass, or the highest points of an anchor ring lying on its side. Most objects intended to stand on a table are provided with a rim on which to sit; they touch the table all along this rim, but do not pierce through the table.

(3) *The surface may touch its tangent plane only at the point* (x_0, y_0); *then* $z=f(x, y)$ *is an* **extreme** *at* (x_0, y_0): a **minimum,** if the surface is wholly above the tangent plane near (x_0, y_0); **a maximum,** if the surface is wholly below.

The shape of the clenched fist gives many good illustrations of this type also. Examples of formal algebraic character occur below.

Example 1. For the elliptic paraboloid $z = x^2 + y^2$ the tangent plane at (x_0, y_0, z_0) is

$$z - z_0 = 2\,x_0\,(x - x_0) + 2\,y_0\,(y - y_0),$$

which is horizontal if $2\,x_0 = 2\,y_0 = 0$; this gives $x_0 = y_0 = z_0 = 0$, hence $(x = 0,\ y = 0)$ is the only *critical point.*

At $(x = 0,\ y = 0)$, z has the value 0; for any other values of x and y, $z\,(= x^2 + y^2)$ is surely positive. It follows that z is a minimum at $x = 0$, $y = 0$.

Example 2 In experiments with a pulley block the weight w to be lifted and the pull p necessary to lift it were found in three trials to be (in pounds) $(p_1 = 5,\ w_1 = 20)$, $(p_2 = 9,\ w_2 = 50)$, $(p_3 = 15,\ w_3 = 90)$. Assuming that $p = \alpha w + \beta$, find the values of α and β which make *the sum S of the squares of the errors least.* (Compare Ex. 18, p. 69, and § 121, p. 229.)

Computing p by the formula $\alpha w+\beta$, the three values are $p'_1=20\,\alpha+\beta$, $p'_2 = 50\,\alpha + \beta$, $p'_3 = 90\,\alpha + \beta$. Hence the sum of the squares of the

errors is

$$\begin{aligned} S &= (p'_1 - p_1)^2 + (p'_2 - p_2)^2 + (p'_3 - p_3)^2 \\ &= (20\,\alpha + \beta - 5)^2 + (50\,\alpha + \beta - 9)^2 + (90\,\alpha + \beta - 15)^2. \end{aligned}$$

In order that S be a minimum, we must have

$$\frac{1}{2}\frac{\partial S}{\partial \alpha} = 20\,(20\,\alpha + \beta - 5) + 50\,(50\,\alpha + \beta - 9) + 90(90\,\alpha + \beta - 15) = 0.$$

$$\frac{1}{2}\frac{\partial S}{\partial \beta} = (20\,\alpha + \beta - 5) + (50\,\alpha + \beta - 9) + (90\,\alpha + \beta - 15) = 0.$$

that is, after reduction,

$$\begin{aligned} 1100\,\alpha + 16\,\beta - 190 &= 0, \\ 160\,\alpha + 3\,\beta - 29 &= 0, \end{aligned} \qquad \text{whence} \qquad \begin{aligned} \alpha &= \tfrac{106}{740} = .143, \\ \beta &= \tfrac{1500}{740} = 2.03. \end{aligned}$$

If the usual graph of the values of p and w is drawn, it will be seen that $p = \alpha w + \beta$ represents these values very well for $\alpha = .143$, $\beta = 2.03$ and it is evident from the geometry of the figure that these values render S a minimum, $S = .0545$; for any considerable increase in either α or β very evidently makes S increase. Since this is the only critical point, it surely corresponds to a minimum, for the function S has no singularities.

This conclusion can also be reached by thinking of S as represented by the heights of a surface over an $\alpha\beta$ plane, and considering the section of that surface by the tangent plane at the point just found as in Ex. 3 below; but in this problem the preceding argument is simpler.

It is customary to assume that the values of α and β which make S a minimum are the best compromise, or the "**most probable values**"; hence the most probable formula for p is $p = .143\,w + 2.03$.

The work based on more than three trials is quite similar; the only change being that S has n terms instead of 3 if n trials are made.

Example 3. Find the most economical dimensions for a rectangular bin with an open top which is to hold 500 cu. ft. of grain.

Let x, y, h represent the width, length, and height of the bin, respectively. Then the volume is xyh; hence $xyh = 500$; and the total area z of the sides and bottom is

$$(a) \qquad z = xy + 2\,hy + 2\,hx = xy + \frac{1000}{x} + \frac{1000}{y}.$$

If this area (which represents the amount of material used) is to be a minimum, we must have

$$(b) \qquad \frac{\partial z}{\partial x} = y - \frac{1000}{x^2} = 0, \quad \frac{\partial z}{\partial y} = x - \frac{1000}{y^2} = 0.$$

Substituting from the first of these the value $y = 1000/x^2$ in the second, we find

$$(c) \qquad x - \frac{x^4}{1000} = 0, \text{ whence } x = 0, \text{ or } x = 10.$$

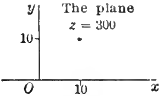


FIG. 71

The value $x = 0$ is obviously not worthy of any consideration; the value $x = 10$ gives $y = 1000/x^2 = 10$ and $h = 500/(xy) = 5$.

The value of z when $x = 10$, $y = 10$ is 300. If the equation (a) is represented graphically by a surface, the values of z being drawn vertical, the section of the surface by the plane $z = 300$ is represented by the equation

$$(d) \qquad xy + \frac{1000}{x} + \frac{1000}{y} = 300, \text{ or } x^2y^2 - 300\,xy + 1000(x + y) = 0.$$

This equation is of course satisfied by $x = 10$, $y = 10$. If we attempt to plot the curve near (10, 10), — for example, if we set $y = 10 + k$ and try to solve for x in the resulting equation :

$$(10 + k)^2x^2 - (300\,k + 2000)x + 1000(10 + k) = 0,$$

the usual rule for imaginary roots of any quadratic $ax^2 + bx + c = 0$ shows that

$$b^2 - 4\,ac = -\,1000\,k^2\,[4\,k + 30] < 0$$

for all values of k greater than $-\,7.5$. Hence it is impossible to find any other point on the curve near (10, 10). It follows that the horizontal tangent plane $z = 300$ cuts the surface in a single point; hence the surface lies entirely on one side of that tangent plane. Trial of any one convenient pair of values of x and y near (10, 10) shows that z is greater near (10, 10) than at (10, 10); hence the area z is a minimum when $x = 10$, $y = 10$, which gives $h = 5$.

166. Final Tests. Final tests to determine whether a function $f(x, y)$ has a maximum or a minimum or neither, are somewhat difficult to obtain in reliable form. Comparatively simple and natural examples are known which escape all set rules of an elementary nature.* (See Example 1 below.)

* For a detailed discussion, see Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 118.

One elementary fact is often useful: if the surface has a maximum at (x_0, y_0), *every vertical section through* (x_0, y_0) *has a maximum there.* Thus any critical point (x_0, y_0) may be discarded if the section by the plane $x = x_0$ has no extreme at that point, or if it has the opposite sort of extreme to the section made by $y = y_0$.

The safest final test, and the one very easy to apply, is to actually draw the section of the surface made by the horizontal tangent plane, as in Ex. 3, § 165. Then a test of a few values quickly settles the matter.

Example 1. The surface $z = (y - x^2)(y - 2x^2)$ has critical points where

$$\frac{\partial f}{\partial x} = -6xy + 8x^3 = 0, \quad \frac{\partial f}{\partial y} = 2y - 3x^2 = 0;$$

that is, the only critical point is $(x = 0, y = 0)$. The tangent plane at that point is $z = 0$. This tangent plane cuts the surface where

$$(y - x^2)(y - 2x^2) = 0;$$

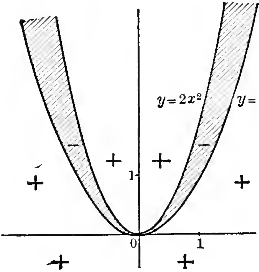


FIG. 72

that is, along the two parabolas $y = x^2$, $y = 2x^2$. At $x = 0$, $y = 1$, the value of z is $+1$; hence z is positive for points (x, y) inside the parabola $y = 2x^2$. At $x = 1$, $y = 0$, the value of z is $+2$; hence z is positive for all points (x, y) outside the parabola $y = x^2$. At the point $x = 1$, $y = 1.5$, the value of z is $-.25$; hence z is negative between the two parabolas. It is evident, therefore, that z has no extreme at $x = 0$, $y = 0$.

A qualitative model of this extremely interesting surface can be made quickly by molding putty or plaster of paris in elevations in the unshaded regions indicated above, with a depression in the shaded portion.

Another interesting fact is that every vertical section of this surface through $(0, 0)$ has a minimum at $(0, 0)$; this fact shows that the rule about vertical sections stated above cannot be reversed. Moreover, this surface eludes every other known elementary test except that used above.

EXERCISES LXVII. — TANGENT PLANES EXTREMES

1. Find the equation of the tangent plane to each of the following surfaces at the point specified :

(*a*) $z = x^2 + 9y^2$, (2, 1, 13). *Ans.* $z = 4x + 18y - 13$.

(*b*) $z = 2x^2 - 4y^2$, (3, 2, 2). *Ans.* $z = 12x - 16y - 2$.

(*c*) $z = xy$, (2, − 3, − 6). *Ans.* $3x - 2y + z = 6$.

(*d*) $z = (x + y)^2$, (1, 1, 4). *Ans.* $4x + 4y - z = 4$.

(*e*) $z = 2xy^2 + y^3$, (2, 0, 0). *Ans.* $z = 0$.

2. The straight line perpendicular to the tangent plane at its point of tangency is called the **normal** to the surface.

Find the normal to each of the surfaces in Ex. 1, at the point specified.

3. At what angle does the plane $x + 2y - z + 3 = 0$ cut the paraboloid $x^2 + y^2 = 4z$ at the point (6, 8, 25) ?

4. Find the angle between the surfaces of Exs. 1 (*a*) and 1 (*b*) at the point ($\sqrt{13}$, 1, 22).

Find the angle between each pair of surfaces in Ex. 1, at some one of their points of intersection, if they intersect.

5. Find the tangent plane to the sphere $x^2 + y^2 + z^2 = 25$ at the point (3, 4, 0) ; at (2, 4, $\sqrt{5}$).

6. At what angles does the line $x = 2y = 3z$ cut the paraboloid $y = x^2 + z^2$.

7. Find a point at which the tangent plane to the surface 1 (*a*) is horizontal.

Draw the contour lines of the surface near that point and show whether the point is a minimum or a maximum or neither.

8. Proceed as in Ex. 7 for each of the surfaces of Ex. 1, and verify the following facts :

(*b*) Horizontal tangent plane at (0, 0) ; no extreme.

(*c*) Horizontal tangent plane at (0, 0) ; no extreme.

(*d*) Horizontal tangent plane at every point on the line $x + y = 0$; weak minimum at each point.

(*e*) Horizontal tangent plane at every point where $y = 0$; no extreme at any point.

9. Find the extremes, if any, on each of the following surfaces:

(*a*) $z = x^2 + 4y^2 - 4x$. (Minimum at $(2, 0, -4)$.)

(*b*) $z = x^3 - 3x - y^2$. (See *Tables*, Fig. I_1.)

(*c*) $z = x^3 - 3x + y^2(x - 4)$. (See *Tables*, Fig. I_2.)

(*d*) $z = [(x-a)^2 + y^2][(x+a)^2 + y^2]$. (Similar to *Tables*, Fig. I_7.)

(*e*) $z = x^3 - 6x - y^2$. (Draw auxiliary curve as for Fig. I_1.)

(*f*) $z = x^3 - 4y^2 + xy^2$. (Draw auxiliary curve as for Fig. I_2.)

(*g*) $z = x^3 + y^3 - 3xy$. (Draw by rotating xy-plane through $\pi/4$.)

10. Redetermine the values of α and β in Example 2, § 165, if the additional information ($p = 23$, $w = 135$) is given.

11. Find the values of u and v for which the expression $(a_1u + b_1v - c_1)^2 + (a_2u + b_2v - c_2)^2 + (a_3u + b_3v - c_3)^2$ becomes a minimum. (Compare Ex. 10.)

12. Show that the most economical rectangular covered box is cubical.

13. Show that the rectangular parallelopiped of greatest volume that can be inscribed in a sphere is a cube.

[Hint. The equation of the sphere is $x^2 + y^2 + z^2 = 1$; one corner of the parallelopiped is at (x, y, z); then $V = 8xyz$, where $z = \sqrt{1 - x^2 - y^2}$.]

14. Show that the greatest rectangular parallelopiped which can be inscribed in an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ has a volume $V = 8abc/(3\sqrt{3})$.

15. The points $(2, 4)$, $(6, 7)$, $(10, 9)$ do not lie on a straight line. Under the assumptions of Ex. 2, § 165, show that the best compromise for a straight line which is experimentally determined by these values is $24y = 15x + 70$.

16. The linear extension E (in inches) of a copper wire stretched by a load W (in pounds) was found by experiment (Gibson) to be ($W = 10$, $E = .06$), ($W = 30$, $E = .17$), ($W = 60$, $E = .32$). Find values of α and β in the formula $E = \alpha W + \beta$ under the assumptions of § 165.

17. The readings of a standard gas meter S and that of a meter T being tested were found to be ($T = 4300$, $S = 500$), ($T = 4390$, $S = 600$), ($T = 4475$, $S = 700$). Find the most probable values in the equation $T = \alpha S + \beta$ and explain the meaning of α and of β.

18. The temperatures $\theta°$ C. at a depth d in feet below the surface of the ground in a mine were found to be $d = 100$ ft., $\theta = 15°.7$, $d = 200$ ft., $\theta = 16°.5$, $d = 300$ ft., $\theta = 17°.4$. Find an expression for the temperature at any depth.

19. Redetermine, under the assumptions of § 165, the most probable values of the constants in Exs. 1–5, p. 236.

20. The points (10, 3.1), (3.3, 1.6), (1.25, .7) lie very nearly on a curve of the form $\alpha/x + \beta/y = 1$. Use the *reciprocals* of the given values to find the most probable values of α and β.

21. The sizes of boiler flues and pressures under which they collapsed were found by Clark to be $(d = 30, p = 76)$, $(d = 40, p = 45)$, $(d = 50, p = 30)$. These values satisfy very nearly an equation of the form $p = k \cdot d^n$ or $\log p = n \log d + \log k$, where d is the diameter in inches, and p is the pressure in pounds per square inch. Using the logarithms of the given numbers, find the most probable values for n and $\log k$.

22. Recompute, under the assumptions of § 165, as in Ex. 21, the values of constants in Exs. 17, 19, pp. 232–233.

167. Tangent Planes. Implicit Forms. If the equation of a surface is given in *implicit form*, $F(x, y, z) = 0$, taking the total differential we find :

$$(1) \qquad \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0.$$

But, by virtue of $F(x, y, z) = 0$, any one of the variables, say z, is a function of the other two; hence

$$(2) \qquad dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Putting this in the total differential above and rearranging:

$$(3) \qquad \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\right)dy = 0.$$

But dx and dy are independent arbitrary increments of x and of y; and since the equation is to hold for all their possible

pairs of values, the coefficients of dx and dy must vanish separately. This gives

$$(4)\qquad \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}, \quad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}.$$

Substituting these values in the equation of the tangent plane, and clearing of fractions, we obtain

$$(5)\qquad \left.\frac{\partial F}{\partial x}\right]_0 (x - x_0) + \left.\frac{\partial F}{\partial y}\right]_0 (y - y_0) + \left.\frac{\partial F}{\partial z}\right]_0 (z - z_0) = 0,$$

the equation of the tangent plane at (x_0, y_0, z_0) to the surface $F(x, y, z) = 0$.

168. Line Normal to a Surface. The direction cosines of the tangent plane to a surface whose equation is given in the *explicit form* $z = f(x, y)$ are proportional (§ 164) to

$$(1)\qquad \partial z/\partial x]_0, \quad \partial z/\partial y]_0, \text{ and } -1.$$

Hence the *equations of the normal* at (x_0, y_0, z_0) are

$$(2)\qquad \frac{x - x_0}{\partial z/\partial x]_0} = \frac{y - y_0}{\partial z/\partial y]_0} = \frac{z - z_0}{-1}.$$

The direction cosines of a surface whose equation is given in the *implicit form* $F(x, y, z) = 0$ are proportional to

$$(3)\qquad \partial F/\partial x]_0, \quad \partial F/\partial y]_0, \quad \partial F/\partial z]_0,$$

so that the equations of the normal to this surface are

$$(4)\qquad \frac{x - x_0}{\partial F/\partial x]_0} = \frac{y - y_0}{\partial F/\partial y]_0} = \frac{z - z_0}{\partial F/\partial z]_0}.$$

169. Parametric Forms of Equations. A surface S may also be represented by expressing the coördinates of any point on it in terms of two auxiliary variables or parameters:

$$[S]\qquad x = f(u, v), \quad y = \phi(u, v), \quad z = \psi(u, v).$$

If we eliminate u and v between these equations, we obtain the equation of the surface in the form $F(x, y, z) = 0$.

Similarly a curve C may be represented by giving x, y, z in terms of a single auxiliary variable or parameter t:

$$[C] \qquad x = f(t), \quad y = \phi(t), \quad z = \psi(t).$$

The elimination of t from each of two pairs of these equations gives the equations of two surfaces on each of which the curve lies, in the form (4), § 163. In particular, taking $t = x$ gives the curve as the intersection of the **projecting cylinders**:

$$[P] \qquad y = \phi(x), \qquad z = \psi(x).$$

If, in the parametric equations of a surface, one parameter (say u) is kept fixed while the other varies, a space-curve is described which lies on the surface. Now if u varies, this curve varies as a whole and describes the surface. The curve on which u keeps the value k is called the curve $u = k$. Similarly, keeping v fixed while u varies gives a curve $v = k'$. The intersection of an $u = k$ with an $v = k'$ gives one or more points (k, k') on the surface. The pair of numbers (k, k') are called the **curvilinear coördinates** of points on the surface.

Simple examples of such coördinates are the ordinary rectangular coördinate system and the polar coördinate system in a plane. Thus $(2, 3)$ means the point at the intersection of the lines $x = 2$, $y = 3$ of the plane; in polar coördinates, $(5, 30°)$ means the point at the intersection of the circle $r = 5$ with the line $\theta = 30°$.

Example 1. The equations of the plane $x + y + z = 1$ may be written, in the parametric form:

$$x = u, \qquad y = v, \qquad z = 1 - u - v.$$

Let the student draw a figure from these equations by inserting arbitrary values of u and v and finding associated values of x, y, z. Another set of parameter equations which represent the same plane is

$$x = u + v, \qquad y = u - v, \qquad z = 0 - 2u + 1.$$

Thus several different sets of parameter equations may represent the same surface.

In the first form, put $u = k$. Then, as v varies, we obtain the straight line

$$x = k, \qquad y = v, \qquad z = 1 - k - v,$$

which lies in the given plane. As k varies this line varies; its different positions map out the entire plane. Likewise, $v = k'$ is a line varying with k' and describing the plane. The intersection of two of these lines, one from each system, is point (k, k') of the plane.

Example 2. The sphere $x^2 + y^2 + z^2 = a^2$ may be represented by the equations:

$$x = a \cos \theta \cos \phi, \qquad y = a \cos \theta \sin \phi, \qquad z = a \sin \theta.$$

Here the parameters θ and ϕ are respectively the latitude and the longitude. Thus $\theta = k$ is a parallel of latitude; $\phi = k'$ is a meridian; and their intersection (k, k') is a point of latitude k and longitude k'. [If a is allowed to vary, the equations of this example define **polar coördinates in space**; but the colatitude $90° - \theta$ is often used in place of θ.]

Example 3. The equations

$$x = a \cos t, \qquad y = a \sin t, \qquad z = bt,$$

represent a space curve, namely a **helix** drawn on a cylinder of radius a with its axis along the z-axis. The total rise of the curve during each revolution is $2\pi b$.

If a is replaced by a variable parameter u, the helix varies with u, and describes the surface

$$x = u \cos t, \qquad y = u \sin t, \qquad z = bt,$$

which is called a **helicoid**. The blade of a propeller screw is a piece of such a surface.

170. Tangent Planes and Normals. Parameter Forms.

When a surface is given by means of *parametric equations*,

$$\text{(1)} \qquad x = f(u, v), \qquad y = \phi(u, v), \qquad z = \psi(u, v),$$

the equation of the tangent plane is found as follows. Elimination of u and v would give the equation in the implicit form $F(x, y, z) = 0$. If the parametric values of x, y, z are substituted in this equation the resulting equation is identically true, since it must hold for all values of the independent parameters u, v; hence

$$\text{(2)} \qquad \frac{\partial F}{\partial u} = 0, \text{ and } \frac{\partial F}{\partial v} = 0,$$

that is

$$\text{(3)} \qquad \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial u} = 0, \qquad \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial v} = 0.$$

Solving these, we find:

$$(4)\qquad \frac{\partial F}{\partial x}:\frac{\partial F}{\partial y}:\frac{\partial F}{\partial z}=\begin{vmatrix}\dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u}\\ \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v}\end{vmatrix}:\begin{vmatrix}\dfrac{\partial z}{\partial u} & \dfrac{\partial x}{\partial u}\\ \dfrac{\partial z}{\partial v} & \dfrac{\partial x}{\partial v}\end{vmatrix}:\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u}\\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v}\end{vmatrix};$$

hence the equation of the tangent plane is

$$(x-x_0)\begin{vmatrix}\dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u}\\ \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v}\end{vmatrix}_0+(y-y_0)\begin{vmatrix}\dfrac{\partial z}{\partial u} & \dfrac{\partial x}{\partial u}\\ \dfrac{\partial z}{\partial v} & \dfrac{\partial x}{\partial v}\end{vmatrix}_0+(z-z_0)\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u}\\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v}\end{vmatrix}_0=0;$$

while the equations of the normal are

$$\frac{x-x_0}{\begin{vmatrix}\dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u}\\ \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v}\end{vmatrix}_0}=\frac{y-y_0}{\begin{vmatrix}\dfrac{\partial z}{\partial u} & \dfrac{\partial x}{\partial u}\\ \dfrac{\partial z}{\partial v} & \dfrac{\partial x}{\partial v}\end{vmatrix}_0}=\frac{z-z_0}{\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u}\\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v}\end{vmatrix}_0}.$$

EXERCISES LXVIII. — EQUATIONS NOT IN EXPLICIT FORM

1. Determine the tangent plane and the normal to the ellipsoid $x^2+4y^2+z^2=36$ at the point $(4, 2, 2)$, first by solving for z, by the methods of § 164; then, without solving for z, by the methods of §§ 167–168.

2. Determine the tangent planes and the normals to each of the following surfaces, at the points specified:

(*a*) $x^2+y^2+z^2=a^2$ at (x_0, y_0, z_0).

(*b*) $x^2-4y^2+z^2=36$ at $(6, 1, 2)$.

(*c*) $x^2-4y^2-9z^2=36$ at $(7, 1, 1)$.

(*d*) $x^2+y^2-z^2=0$ at $(3, 4, 5)$.

(*e*) $x^3+x^2y-2z^2=0$ at $(1, 1, -1)$.

(*f*) $z^2=e^{x+y}$ at $(0, 0, 1)$.

3. Find the angle between the tangent planes to the ellipsoid $4x^2+9y^2+36z^2=36$ at the points $(2, 1, z_0)$ and $(-1, -1, z_1)$.

4. At what angle does the z-axis cut the surface $z^2=e^{x+y}$?

5. Obtain the equation of the tangent plane to the helicoid

$$x = u \cos v, \qquad y = u \sin v, \qquad z = v,$$

at the point $u = 1$, $v = \pi/4$.

6. Taking the equations of a sphere in terms of the latitude and longitude (Example 2, § 169), find the equation of its tangent plane and the equations of the normal at a point where $\theta = \phi = 45°$; at a point where $\theta = 60°$, $\phi = 30°$.

7. Eliminate u and v from the equations $x = u + v$, $y = u - v$, $z = uv$, to obtain an equation in x, y, and z. Find the equation of the tangent plane at a point where $u = 3$, $v = 2$, by the methods of § 164; then by the methods of § 170 directly from the given equations.

8. Write the equation of the tangent plane to the surface used in Ex. 7 at any point (x_0, y_0, z_0). At what point is the tangent plane horizontal? Is z an extreme at that point?

9. Proceed as in Ex. 7 for each of the following surfaces:

(a) $x = r \cos \theta$, $y = r \sin \theta$, $z = r$, at $r = 2$, $\theta = \pi/4$.

(b) $x = \dfrac{uv + 1}{u + v}$, $y = \dfrac{u - v}{u + v}$, $z = \dfrac{uv - 1}{u + v}$, at $u = 2$, $v = -1$.

(c) $x = -3u + 2v$, $y = 2u - v$, $z = e^{u+v}$, at (u_0, v_0).

(d) $x = 2 \cos \theta \cos \phi$, $y = 3 \cos \theta \sin \phi$, $z = \sin \theta$, at $\theta = \phi = \pi/4$.

10. The surfaces $z = x^2 - 4y^2$ and $z = 6x$ intersect in a curve, whose equations are the two given equations. Find the tangent line to this curve at the point (8, 2, 48) by first finding the tangent planes to each of the surfaces at that point; the line of intersection of these planes is the required line.

11. Find the tangent line to the curve defined by the two equations $16x^2 - 3y^2 = 4z$ and $9x^2 + 3y^2 - z^2 = 20$ at (1, 2, 1).

171. Area of a Curved Surface. Let S be a portion of a curved surface and R its projection on the xy-plane. In R take an element $\Delta x \Delta y$ and on it erect a prism cutting an element ΔS out of S. At any point of ΔS, draw a tangent plane. The prism cuts from this an element ΔA. The smaller $\Delta x\, \Delta y$ (and therefore ΔS) becomes, the more nearly will the ratio $\Delta A/\Delta S$ approach unity, since the limit of this ratio is 1.

Suppose now that the area R is all divided up into elements $\Delta x \Delta y$ and

that on each a prism is erected. The area S will thus be divided up into elements ΔS and there will be cut from the tangent plane at a point of each an element ΔA. One thus gets

(1) $$S = \lim_{\substack{\Delta x \doteq 0 \\ \Delta y \doteq 0}} \sum \Delta A.$$

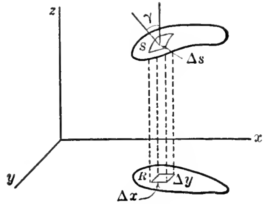


Fig. 73

But if γ is the acute angle that the normal to any ΔA makes with the z-axis, we have

(2) $$\Delta A = \sec\gamma\, \Delta x\, \Delta y;$$

hence

(3) $$S = \lim_{\substack{\Delta x \doteq 0 \\ \Delta y \doteq 0}} \sum \Delta A = \lim_{\substack{\Delta x \doteq 0 \\ \Delta y \doteq 0}} \sum (\sec\gamma\, \Delta x\, \Delta y) = \int\int_R \sec\gamma\, dx\, dy.$$

Of course $\sec\gamma$ is a variable to be expressed in terms of x and y from the equation of the surface. The limits of integration to be inserted are the same as if the area of R were to be found by means of the integral

$$\int\int dx\, dy.$$

If the surface doubles back on itself, so that the projecting prisms cut it more than once, it will usually be best to calculate each piece separately.

When the equation of the surface is given in the form $z = f(x, y)$, the direction cosines of the normal are given by

$$\cos\alpha : \cos\beta : \cos\gamma = \frac{\partial z}{\partial x} : \frac{\partial z}{\partial y} : -1.$$

Taking $\cos\gamma$ positive, that is γ acute, we may write

(4) $$\sec\gamma = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1},$$

and $$S = \int\int \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dx\, dy.$$

The determination of $\sec\gamma$, when the surface is given in the form $F(x, y, z) = 0$, is performed by straightforward transformations similar to those used in §§ 167–170; they are left to the student.

EXERCISES LXIX. — AREA OF A SURFACE

1. Calculate the area of a sphere by the preceding method.

2. A square hole is cut centrally through a sphere. How much of the spherical surface is removed?

3. A cylinder intersects a sphere so that an element of the cylinder coincides with a diameter of the sphere. If the diameter of the cylinder equals the radius of the sphere, what part of the spherical surface lies within the cylinder?

4. How much of the surface $z = xy$ lies within the cylinder $x^2 + y^2 = 1$?

5. How much of the conical surface $z^2 = x^2 + y^2$ lies above a square in the xy-plane whose center is the origin?

6. Show that if the region R of § 171 be referred to ordinary polar coördinates, $\Delta A = r \sec\gamma\, \Delta r\, \Delta\theta$, approximately. (See [B], p. 212.)

7. Using the result of Ex. 6, show that $S = \int\int r \sec\gamma\, dr\, d\theta$.

8. Show that, for a surface of revolution formed by revolving a curve whose equation is $z = f(x)$ about the z axis, $\sec\gamma = \sqrt{1 + [df(r)/dr]^2}$, where $r = \sqrt{x^2 + y^2}$.

9. By means of Exs. 7, 8, show that the area of the surface of revolution mentioned in Ex. 8 is

$$S = \int_0^a \int_0^{2\pi} r\sqrt{1 + \left[\frac{df(r)}{dr}\right]^2}\, dr\, d\theta = 2\pi \int_0^a r\sqrt{1 + \left[\frac{df(r)}{dr}\right]^2} dr,$$

where a is the value of r at the end of the arc of the generating curve. (See Ex. 13, p. 129.)

10. Compute the area of a sphere by the method of Ex. 9.

11. Find the area of the portion of the paraboloid of revolution formed by revolving the curve $z^2 = 2mx$ about the x axis, from $x = 0$ to $x = k$.

12. Show that the area of the surface of an ellipsoid of revolution is $2\pi b\,[b + (a/e)\sin^{-1}e]$, where a and b are the semiaxes and e the eccentricity, of the generating ellipse.

13. Show that the area generated by revolving one arch of a cycloid about its base is $64\pi a^2/3$.

14. Show that the area of the surface generated by revolving the curve $x^{2/3} + y^{2/3} = a^{2/3}$ about one of the axes is $12\pi a^2/5$.

172. Tangent to a Space Curve. Let the equation of the curve be given in parametric form $x = f(t)$, $y = \phi(t)$, $z = \psi(t)$. Let $P_0 = (x_0, y_0, z_0)$ be the point on the curve where $t = t_0$. Let Q be a neighboring point on the curve where $t = t_0 + \Delta t$.

The direction cosines of the secant P_0Q are proportional to $\Delta x/\Delta t$, $\Delta y/\Delta t$, $\Delta z/\Delta t$; hence its equations are

$$(1)\quad \frac{x - x_0}{\Delta x/\Delta t} = \frac{y - y_0}{\Delta y/\Delta t} = \frac{z - z_0}{\Delta z/\Delta t}.$$

As $\Delta t \doteq 0$, these become

$$(2)\quad \frac{x - x_0}{dx/dt]_0} = \frac{y - y_0}{dy/dt]_0} = \frac{z - z_0}{dz/dt]_0},$$

the equations of the tangent at the point P_0.

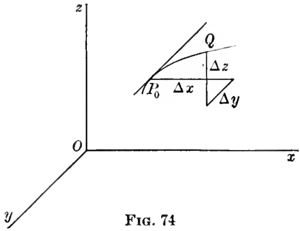


FIG. 74

If the curve is given as the intersection of two projecting cylinders $y = f(x)$, $z = \phi(x)$, we may join to these the third equation $x = x$, thus conceiving of x, y, and z as all expressed in terms of x. The equations of the tangent then become

$$(3)\qquad \frac{x - x_0}{1} = \frac{y - y_0}{dy/dx]_0} = \frac{z - z_0}{dz/dx]_0}.$$

If the curve is given as the intersection of two surfaces, $f(x, y, z) = 0$, $F(x, y, z) = 0$, and if we think of x, y, z as depending upon a parameter t, we find

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0,$$

and

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0.$$

From these equations we obtain $dx/dt : dy/dt : dz/dt$, and we may write the equations of the tangent at P_0 in the form:

$$\frac{x - x_0}{\begin{vmatrix} \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z} \end{vmatrix}_0} = \frac{y - y_0}{\begin{vmatrix} \dfrac{\partial f}{\partial z} & \dfrac{\partial f}{\partial x} \\ \dfrac{\partial F}{\partial z} & \dfrac{\partial F}{\partial x} \end{vmatrix}_0} = \frac{z - z_0}{\begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \end{vmatrix}_0}.$$

173. Length of a Space Curve. The length of the chord joining two points t and $t + \Delta t$ of the curve

(1) $$x = f(t),\ y = \phi(t),\ z = \psi(t),$$

is $\Delta c = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$, or,

(2) $$\Delta c = \sqrt{\frac{\Delta x^2}{\Delta t^2} + \frac{\Delta y^2}{\Delta t^2} + \frac{\Delta z^2}{\Delta t^2}}\Delta t.$$

Defining the length of a curve between two points as the limit of the sum of the inscribed chords (see § 12, p. 18), we find for that length:

(3) $$s = \lim_{\Delta t \doteq 0} \sum \Delta c = \int_{t=t_0}^{t=t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

EXERCISES LXX. — TANGENTS TO CURVES LENGTHS

1. Write the equation of the tangent at an arbitrary point of each of the curves in Ex. 12, p 320.

2. At what angle does a straight line joining the earth's South pole with a point in 40° North latitude cut the 40th parallel?

3. At what angle does the helix $x = 2\cos\theta,\ y = 2\sin\theta,\ z = \theta$, cut the sphere $x^2 + y^2 + z^2 = 9$?

4. Find the angle of intersection of the ellipse and parabola that are cut from the cone $z^2 = x^2 + y^2$ by the planes $2z = 1 - x$ and $z = 1 + x$ respectively.

5. Show that the curves of intersection of the three surfaces

$$z = y,\ x^2 = y^2 + z^2,\ x^2 + y^2 + z^2 = 1,$$

cut each other mutually at right angles.

6. Show the same for the curves of intersection on the surfaces

$$4x^2 + 9y^2 + 36z^2 = 36,\ 3x^2 + 6y^2 - 6z^2 = 6,\ 10x^2 - 15y^2 - 6z^2 = 30.$$

7. Calculate the length of the curve $x = t,\ y = t^2,\ z = 2t^{3/2}$, from $t = 0$ to $t = 1$.

8. Find the length of the helix $x = a\cos\theta,\ y = a\sin\theta,\ z = b\theta$, from $\theta = \theta_0$ to $\theta = \theta_1$. What is the length of one turn?

9. Find the length of the curve $x = \sin z,\ y = \cos z$, from $(1, 0, \pi/2)$ to $(0, -1, \pi)$.

EXERCISES LXXI. — GENERAL REVIEW SEVERAL VARIABLES

[The exercises marked with an asterisk are of more than usual difficulty. Some of them contain new concepts of value for which it is hoped that time may be found. Those of the greatest theoretical value are marked †.

Attention is called to the reviews of double and triple integration.]

1. Given $u = xy$, $x = r\cos\theta$, $y = r\sin\theta$, find $\partial u/\partial r$ and $\partial u/\partial\theta$, first by actually expressing u in terms of r and θ; then directly from the given equations.

2. Proceed as in Ex. 1 for the function $u = \tan^{-1}(y/x)$.

3. Given $u = r^2 e^{2\theta}$, $x = r\cos\theta$, $y = r\sin\theta$, find $\partial u/\partial x$ and $\partial u/\partial y$, first by expressing u in terms of x and y; then directly from the given equations.

[HINT. In the second part, it is convenient here to solve the last two equations for r and θ in terms of x and y. But see Ex. 4.]

4.* If $x = r\cos\theta$ and $y = r\sin\theta$, show by differentiation that

$$\frac{\partial x}{\partial x} = 1 = \frac{\partial r}{\partial x}\cos\theta - r\sin\theta\frac{\partial\theta}{\partial x}, \quad \text{and} \quad \frac{\partial y}{\partial x} = 0 = \frac{\partial r}{\partial x}\sin\theta + r\cos\theta\frac{\partial\theta}{\partial x}.$$

Solve these equations for $\partial r/\partial x$ and $\partial\theta/\partial x$, and show that $\partial u/\partial x$ may be found in Ex. 3 by means of the equation

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x}.$$

5.* If, in general, u is a function of the two variables (r, θ), show that the last equation in Ex. 4 holds true. Find a similar equation for $\partial u/\partial y$, and evaluate $\partial u/\partial y$ in Ex. 3 by means of it.

6.*† If u is a function of any two variables p and q, and if p and q are given in terms of x and y by two equations $x = f(p, q)$, $y = \phi(p, q)$, obtain $\partial u/\partial x$ and $\partial u/\partial y$ by a process analogous to that of Exs. 4, 5.

7. Proceed as in Ex. 3, by the methods of Exs. 4, 5, in each of the following cases:

$$(a)\ u = r^2 - \cos^2\theta, \qquad (b)\ u = re^{\theta^2}, \qquad (c)\ u = \theta\log r.$$

8. Find the volume of that portion of a sphere of radius 4 ft. which is bounded by two parallel planes at distances 2 ft. and 3 ft., respectively, from the center, on the same side of the center.

9. Determine the position of the center of mass of the solid described in Ex. 8.

10. What is the nature of the field of integration in the integral

$$\int_0^{a/\sqrt{2}}\int_x^{\sqrt{a^2-x^2}} f(x, y)\, dy\, dx\,?$$

Show that the same integral may be written in the form

$$\int_0^{a\sqrt{2}}\int_0^{x} f(x, y)\, dy\, dx + \int_{a/\sqrt{2}}^{a}\int_0^{\sqrt{a^2-y^2}} f(x, y)\, dx\, dy.$$

11. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 - ax = 0$.

12. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $(x - a)^2 + y^2 - z^2 = 0$.

13. Show that the surface of a zone of a sphere depends only upon the radius of the sphere and the height $b - a$ of the zone, where the bounding planes are $z = a$ and $z = b$.

14. Find the area of that part of the surface $k^2 z = xy$ within the cylinder $x^2 + y^2 = k^2$.

15. Find the center of gravity of the portion of the surface described in Ex. 14, when $k = 1$.

16. Find the moment of inertia about its edge, of a wedge whose cross section, perpendicular to the edge, is a sector of a circle of radius 1 and angle 30°, if the length of the edge is 1, and the density is 1.

17. The thrust due to water flowing against an element of a surface is proportional to the area of the element and to the square of the component of the speed perpendicular to the element. Show that the total thrust on a cone whose axis lies in the direction of the flow is $k\pi r^3 v^2/(r^2+h^2)^{\frac{1}{2}}$.

18. Calculate the total thrust due to water flowing against a segment of a paraboloid of revolution whose axis lies in the direction of the flow. (See Ex. 17.)

19. Show that the thrust due to water flowing against a sphere is $2\,k\pi r^2 v^2/3$. Compare with the thrust due to the flow normally against a diametral plane of this sphere.

20.*† Given a function $f(x, y)$, consider the function

$$\phi(t) = f(a + ht,\ b + kt),$$

and show, by means of Maclaurin's series for $\phi(t)$ [see [D*]′, § 134, p. 258,] that, upon inserting the special value $t = 1$, we obtain:

$$f(a + h,\ b + k) = f(a,\ b) + \left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right]_{\substack{x=a\\y=b}} +$$

$$+ \frac{1}{2\,!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2\,hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right]_{\substack{x=a\\y=b}} + \cdots$$

$$+ \frac{1}{(n-1)\,!}\left[h^{n-1}\frac{\partial^{n-1} f}{\partial x^{n-1}} + (n-1)\,h^{n-2}k\frac{\partial^{n-1} f}{\partial x^{n-2}\partial y} + \cdots\right]_{\substack{x=a\\y=b}} + E_n,$$

where $|E_n| \leqq M(|h| + |k|)^n \div n\,!$, and where M is the maximum of the absolute values of all the nth derivatives in a rectangle whose sides are $x = a$, $x = a + h$, $y = b$, $y = b + k$. [TAYLOR'S THEOREM.]

21.† Assuming the truth of the formula of Ex. 20, show that the special values $a = 0$, $b = 0$, $h = x$, $k = y$, lead to the formula

$$f(x,\ y) = f(0,\ 0) + \left[x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right]_{(0,\ 0)} + \cdots + E_n.$$

22. Expand each of the following functions by use of the formula of Ex. 21, in powers of x and y as far as terms of the second degree:

(a) $\sin(x + y)$. (b) e^{2x+3y}. (c) $\cos(x^2 + y^2)$.

23. Find the critical points, if any exist, for the surface $z = x^2 + 2\,y^2 - 4\,x - 4\,y + 10$. Is the value of z an extreme at that point? Draw the contour lines near the point.

24. Determine the greatest rectangular parallelopiped which can be inscribed in a sphere of radius a.

25. The volume of CO_2 dissolved in a given amount of water at temperature θ is

$$\begin{cases} \theta & 0 \quad 5 \quad 10 \quad 15, \\ v & 1.80 \quad 1.45 \quad 1.18 \quad 1.00. \end{cases}$$

Determine the most probable relation of the form $v = a + b\theta$.

26. Determine the most probable relation of the form $S = a + bP^2$ from the data:

$$\begin{cases} P & 550 \quad 650 \quad 750 \quad 850, \\ S & 26 \quad 35 \quad 52 \quad 70. \end{cases}$$

27. Determine the most probable relation of the form $y = ae^{bx}$ from the data:

$$\begin{cases} x & 1 \quad 2 \quad 3 \quad 4, \\ y & .74 \quad .27 \quad .16 \quad .04. \end{cases}$$

28. The barometric pressure P (inches) at height H (thousands feet) is

$$\begin{cases} P & 30 & 28 & 26 & 24 & 22 & 20 & 18 & 16, \\ H & 0 & 1.8 & 3.8 & 5.9 & 8.1 & 10.5 & 13.2 & 16.0. \end{cases}$$

Determine the most probable values of the constants in each of the assumed relations: (*a*) $H = a + bP$; (*b*) $H = a + bP + cP^2$; (*c*) $H = a + b \log P$ or $P = Ae^{BH}$. Which is the best approximation?

29†. If the observed values of one quantity y are m_1, m_2, m_3, corresponding to values l_1, l_2, l_3 of a quantity x on which y depends, and if $y = ax + b$, show that the sum

$$S = (al_1 + b - m_1)^2 + (al_2 + b - m_2)^2 + (al_3 + b - m_3)^2$$

is least when

$$\begin{cases} l_1(al_1 + b - m_1) + l_2(al_2 + b - m_2) + l_3(al_3 + b - m_3) = 0, \\ (al_1 + b - m_1) + (al_2 + b - m_2) + (al_3 + b - m_3) = 0; \end{cases}$$

that is, when

$$a \cdot \sum l_1^2 + b \cdot \sum l_1 - \sum m_1 l_1 = 0 \text{ and } a \cdot \sum l_1 + 3b - \sum m_1 = 0,$$

or

$$a = \frac{3\sum m_1 l_1 - \sum m_1 \cdot \sum l_1}{3\sum l_1^2 - \left[\sum l_1\right]^2}, \quad b = \frac{\sum l_1^2 \cdot \sum m_1 - \sum m_1 l_1 \cdot \sum l_1}{3\sum l_1^2 - \left[\sum l_1\right]^2},$$

where $\sum$ indicates the sum of such terms as that which follows it.

[Theory of Least Squares.]

30. Show that the equation of the tangent plane to $2z = x^2 + y^2$ at (x_0, y_0) is $z + z_0 = xx_0 + yy_0$.

31. Determine the tangent plane and normal line to the hyperboloid $x^2 - 4y^2 + 9z^2 = 36$ at the point $(2, 1, 2)$.

32. Study the surface $xyz = 1$. Show that the volume included between any tangent plane and the coördinate planes is constant.

33. Study the surface $z = (x^2 + y^2)(x^2 + y^2 - 1)$. Determine the extremes.

34. At what angle does a line through the origin and equally inclined to the positive axes cut the surface $2z = x^2 + y^2$?

35. Determine the tangent line and the normal plane at the point $(1, 3/8, 5/8)$ on the curve of intersection of the surfaces $x + y + z = 2$ and $x^2 + 4y^2 - 4z^2 = 0$.

36. Determine the tangent line and the normal plane to the curve $x = 2\cos t$, $y = 2\sin t$, $z = t^2$ at $t = \pi/2$ and at $t = \pi$.

37. Find the length of one turn of the conical spiral $x = t\cos(a\log t)$, $y = t\sin(a\log t)$, $z = bt$, starting from $t = t$.

38. Determine the length of the curve $x = a\cos\theta\cos\phi$, $y = a\sin\theta\cos\phi$, $z = a\sin\phi$, from $\phi = \phi_1$ to $\phi = \phi_2$, where θ is given in terms of ϕ by the equation $\theta = k\log\cot(\pi/4 - \phi/2)$. (Loxodrome on the sphere.)

39.*† Show that the surfaces $f(x, y, z) = 0$ and $\phi(x, y, z) = 0$ cut each other at right angles if $f_x\phi_x + f_y\phi_y + f_z\phi_z = 0$.

40.* Show that the surfaces

$$x^2/(a^2+\lambda) + y^2/(b^2+\lambda) + z^2/(c^2+\lambda) = 1, \qquad a>b>c>0,$$

are always (*i*) ellipsoids if $\lambda > -c^2$, (*ii*) hyperboloids of one sheet if $-b^2 < \lambda < -c^2$, (*iii*) hyperboloids of two sheets if $-a^2 < \lambda < -b^2$.

(Confocal Quadrics.)

Show also that these surfaces cut each other mutually at right angles.

41.*† On the surface $x = f(u, v)$, $y = \phi(u, v)$, $z = \psi(u, v)$, show that ds (or $\sqrt{dx^2+dy^2+dz^2}$) $= \sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}$, where $E = f_u^2 + \phi_u^2 + \psi_u^2$, $F = f_u f_v + \phi_u\phi_v + \psi_u\psi_v$, $G = f_v^2 + \phi_v^2 + \psi_v^2$.

42. If $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$, $z = r\sin\theta$ (polar coördinates), find $\partial u/\partial r$, $\partial u/\partial\theta$, and $\partial u/\partial\phi$ for each of the following functions:

(*a*) $u = x^2 + y^2 + z^2$, (*b*) $u = x^2 + y^2 - z^2$, (*c*) $u = ze^{x+y}$.

43. Compute $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ if $u = r^2(\sin^2\theta + \sin^2\phi)$, where r, θ, ϕ are defined as in Ex. 42.

44.* If (r, θ) are the polar coördinates of a point in the plane whose rectangular coördinates are (x, y), and if u is any function of x and y, show that

$$u_x \equiv \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}$$

$$u_y \equiv \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}.$$

45.† Show that the centroid $(\bar{x}, \bar{y})$ of a plane area in polar coördinates (ρ, θ) is

$$\bar{x} = \frac{\int\int \rho^2\cos\theta\,d\rho\,d\theta}{\int\int \rho\,d\rho\,d\theta}, \quad \bar{y} = \frac{\int\int \rho^2\sin\theta\,d\rho\,d\theta}{\int\int \rho\,d\rho\,d\theta},$$

where the integrals are extended over the given area.

46.* Repeat the process of Ex. 44 on the functions u_x and u_y to show that

$$u_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial r}\cos\theta - \frac{\partial u_x}{\partial\theta}\frac{\sin\theta}{r}$$

$$u_{yy} = \frac{\partial u_y}{\partial r}\sin\theta + \frac{\partial u_y}{\partial\theta}\frac{\cos\theta}{r};$$

and by means of the relations

$$\frac{\partial u_x}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right) = \frac{\partial^2 u}{\partial r^2}\cos\theta + \frac{\partial u}{\partial\theta}\frac{\sin\theta}{r^2} - \frac{\partial^2 u}{\partial r\,\partial\theta},$$

$$\frac{\partial u_x}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right) = \frac{\partial^2 u}{\partial r\,\partial\theta}\cos\theta - \frac{\partial u}{\partial r}\sin\theta - \frac{\partial^2 u}{\partial\theta^2}\frac{\sin\theta}{r} - \frac{\partial u}{\partial\theta}\frac{\cos\theta}{r},$$

and similar relations for $\partial u_y/\partial r$ and $\partial u_y/\partial\theta$, show that

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial u_x}{\partial x} = \frac{\partial^2 u}{\partial r^2}\cos^2\theta - \frac{\partial^2 u}{\partial r\,\partial\theta}\frac{2\sin\theta\cos\theta}{r} + \frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r} + \frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r},$$

and

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial u_y}{\partial y} = \frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{\partial^2 u}{\partial r\,\partial\theta}\frac{2\sin\theta\cos\theta}{r} + \frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2} + \frac{\partial v}{\partial r}\frac{\cos^2\theta}{r} - \frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r}.$$

47.*† By means of Ex. 46, show that **Laplace's Equation**, $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$, is equivalent, in polar coördinates, to the equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0.$$

48. Show that $u = \log r$ is a solution of Laplace's equation (Ex. 47) by direct substitution in the last equation.

49.*† Complete differentials. If $u = f(x, y)$ we know that

$$du = (\partial f/\partial x)\,dx + (\partial f/\partial y)\,dy;$$

hence if du is given, say $du = P(x, y)dx + Q(x, y)dy$, we have $P = \partial f/\partial x$, $Q = \partial f/\partial y$. Show that $\partial P/\partial y = \partial Q/\partial x$.

50. Given $du = (e^x + \sin y)\,dx + (e^y + x\cos y)dy$, show that the condition $\partial P/\partial y = \partial Q/\partial x$ of Ex. 49 is satisfied. [See also Exs. 4–11, p. 361.]

CHAPTER X

DIFFERENTIAL EQUATIONS

PART I. ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

174. Reversal of Rates. In Chapter V we studied the problem of *finding a function whose derivative is given, i.e.* the problem of reversed rates. We found that if

(1) $$\frac{dy}{dx} = f(x)$$

is given, the original function y can be found:

$$y = \int f(x)\, dx + C,$$

at least to within an arbitrary constant.

175. Other Reversed Problems. The preceding process was applied in various ways. Thus we found (§ 113, p. 206) that the distance s passed over by a moving body could be found if the speed v is given:

(1) $$s = \int v\, dt + c; \quad v = \frac{ds}{dt} = f(t).$$

A similar process was applied repeatedly: thus we found the speed v from the tangential acceleration:

(2) $$v = \int j_T dt + C; \quad j_T = \frac{dv}{dt} = \phi(t);$$

hence

(3) $$s = \int \left\{ \int j_T\, dt + C_1 \right\} dt + C_2 = \int \int j_T\, dt\, dt + C_1 t + C_2$$

Again, in (§ 83, p. 147), we found that $(dy/dx) \div y$ expresses the relative rate of change (logarithmic derivative); and we saw that the only function whose relative rate is constant is a compound interest function:

$$(4) \qquad \frac{dy}{dx} \div y = k \ \textit{ gives } \ \log_e y = kx + c, \text{ or } y = Ce^{kx},$$

where $C = e^c$.

Finally, in § 92, p. 162, we found that a damped vibration of the form $y = e^{-ax} \sin (kx + \epsilon)$ satisfies a differential equation of the form

$$(5) \qquad \frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + (k^2 + a^2)y = 0.$$

176. Determination of the Arbitrary Constants. The determination of the arbitrary constants appeared in the very first examples. Thus, in § 54, p. 91, the rate at which water is being poured into a tank was considered. The total amount y was found to be

$$y = r \cdot t + C,$$

where r is the rate per second, t is the time in seconds, and C is the amount already in the tank when $t = 0$.

The arbitrary constant C is determined as soon as the value of y is given for some value of x. Thus in the problem of falling bodies (§ 113, p. 206), from the fact that

$$j_T = -\text{ const.} = -g = -32.16 \text{ ft./sec./sec.},$$

we found that

$$s = -\tfrac{1}{2}gt^2 + c_1 t + c_2.$$

If the body is dropped from rest ($v = 0$), at a height $s = 100$ ft., we have

$$s\Big]_{t=0} = c_2 = 100, \quad v\Big]_{t=0} = c_1 = 0,$$

whence

$$s = -\tfrac{1}{2}gt^2 + 0 + 100,$$

in which the arbitrary constants have disappeared.

Essentially the same process was used in determining the arbitrary constants in a compound interest law (§ 81, p. 143).

Finally, in the case of direct integration, the arbitrary constant was disposed of by taking the difference between two values of y which correspond to two given values of x:

$$y\Big]_{x=a}^{x=b} = \int_{x=a}^{x=b} f(x)\,dx; \quad \frac{dy}{dx} = f(x), \text{ given};$$

and the same scheme was used in motion problems (§ 59, p. 100) and in compound interest examples (§ 81, p. 142).

177. Vital Character of Inverse Problems. These problems are reversed or indirect only from a mathematical standpoint. From the standpoint of science, or of everyday life, many such problems are more direct than those which seem to be the original ones from a mathematical standpoint.

Thus, from the standpoint of science, it is just as much a direct problem to find the distance passed over from a given acceleration, as to find the acceleration from the distance; as a matter of fact the former is usually the *real scientific problem.*

We found (§ 97, p. 169,) that the radius of curvature of any given curve is $[1 + m^2]^{3/2}/b$, where $m = dy/dx$, $b = d^2y/dx^2$. If the curve is given, this formula indeed gives the radius of curvature. But it is more desirable in practice to find a curve whose radius of curvature behaves in a way we wish: given the radius of curvature $R = \psi(x)$, it is desired to find a curve $y = f(x)$ which will actually have just this radius at each point:

(1) $$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \div \frac{d^2y}{dx^2} = \psi(x).$$

We shall solve such differential equations later (§ 191, p. 371); just now it is important to see that they actually arise in concrete direct scientific and mathematical problems.

178. Elementary Definitions. Ordinary Differential Equations. An **ordinary** differential equation is one involving only one independent variable. The derivatives in such an equation are therefore ordinary derivatives.

An ordinary differential equation may contain derivatives of various orders, and these derivatives may enter in various powers.

The **order** of a differential equation is the order of the highest derivative present in it.

The **degree** of a differential equation is the exponent of the highest power of the highest derivative, the equation having been made rational and integral in the derivatives which occur in it.

Thus, equation (5), § 175, is of the second order and first degree; (1), § 174; (1), § 175; (4), § 175, are of the first order and first degree; and (1), § 177, when rationalized, is of the second order and second degree.

179. Elimination of Constants. Differential equations also arise in the elimination of arbitrary constants from an equation.

Example 1. Thus, if A and B are arbitrary constants, the equation $y = Ax + B$ represents a straight line in the plane, and by a proper choice of A and B represents any line one pleases in the plane except a vertical line. One differentiation gives $m = dy/dx = A$, which represents all lines of slope A. A second differentiation gives

$$\text{(1)} \qquad \textit{flexion} = b = d^2y/dx^2 = 0,$$

which represents all non-vertical lines in the plane, since all these and no other curves have a flexion identically zero.

Example 2. Any circle whose radius is a given constant r is represented by the equation

$$(x - A)^2 + (y - B)^2 = r^2, \tag{2}$$

from which A and B may be eliminated as in the preceding example. Differentiating once,

$$x - A + (y - B)y' = 0, \tag{3}$$

where $y' = dy/dx$. Differentiating again,

$$1 + y'^2 + (y - B)y'' = 0, \tag{4}$$

where $y'' = d^2y/dx^2$. Solving (3) and (4) for $x - A$ and $y - B$ and substituting these values into (2), A and B are eliminated, giving

$$(1 + y'^2)^3 = r^2y''^2. \tag{5}$$

This says that every one of these circles, regardless of the position of its center, has the curvature $1/r$, — a statement which absolutely characterizes these circles.

In general, if

$$f(x, y, c_1, c_2, \cdots, c_n) = 0 \tag{6}$$

is an equation involving x, y, and n independent arbitrary constants $c_1, c_2, \cdots, c_n$, n differentiations in succession with regard to x give

$$\frac{df}{dx} = 0, \quad \frac{d^2f}{dx^2} = 0, \quad \cdots, \quad \frac{d^nf}{dx^n} = 0; \tag{7}$$

these equations, together with (6), form a system of $n + 1$ equations from which the constants $c_1, c_2, \cdots, c_n$ may be eliminated. The result is a differential equation of the nth order free from arbitrary constants, and of the form

$$\phi(x, y, y', y'', \cdots, y^{(n)}) = 0. \tag{8}$$

Equation (6) is called the **primitive** or the **general solution** of (8). The term *general solution* is used because it can be shown that all possible solutions of an ordinary differential equation of the nth order can be produced from any solution that involves n independent arbitrary constants, with the exception of certain so-called "singular solutions" not derivable

from the one general solution (6) (see Ex. 20, List LXXV, p. 362).

Thus, to solve an ordinary differential equation of the nth order is understood to mean to find a relation between the variables and n arbitrary constants. These latter are called the *constants of integration.*

If, in the general solution, particular values are assigned to the constants of integration, a *particular solution* of the differential equation is obtained.

180. Integral Curves. An ordinary differential equation of the first order,

(1) $$\phi(x, y, y') = 0, \quad \text{or} \quad y' = f(x, y),$$

where $y' = dy/dx$, has a general solution involving *one* arbitrary constant c:

(2) $$F(x, y, c) = 0.$$

This represents a singly infinite *set* or *family* of curves, there being in general one curve for each value of c. Any curve of the family can be singled out by assigning to c the proper value.

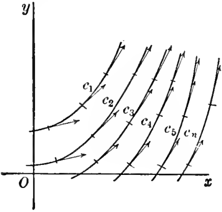


Fig. 75

The differential equation determines these curves by assigning, for each pair of values of x and y, that is, at each point of the plane, a value of the slope y' $[= f(x, y)]$ of the particular curve going through that point. Thus the curves are outlined by the directions of their tangents in much the way that iron filings sprinkled over a glass plate arrange themselves in what seem to the eye to be curves when a magnet is placed beneath the glass. Straws on water in motion create the same optical illusion.

A differential equation of the second order:

$$\phi(x, y, y', y'') = 0, \quad \text{or } y'' = f(x, y, y'),$$

has a general solution involving two arbitrary constants,

$$F(x, y, c_1, c_2) = 0.$$

This represents a *doubly infinite* or *two-parameter family* of curves; for each constant, independently of the other, can have any value whatever. The extension of these concepts to equations of higher order is obvious.

The curves which constitute the solutions are called the **integral curves** of the differential equation.

EXERCISES LXXII. — ELIMINATION INTEGRAL CURVES

Find the differential equations whose general solutions are the following, the c's denoting arbitrary constants:

1. $x^2 + y^2 = c^2$. *Ans.* $x + yy' = 0$.
2. $x^2 - y^2 = cx$. *Ans.* $x^2 + y^2 = 2\,xyy'$.
3. $y = ce^x - \frac{1}{2}(\sin x + \cos x)$. *Ans.* $y' = y + \sin x$.
4. $y = cx + c^2$. *Ans.* $y = y'x + y'^2$.
5. $y = cx + f(c)$. *Ans.* $y = y'x + f(y')$.
6. $y = c_1e^{2x} + c_2e^{3x}$. *Ans.* $y'' - 5\,y' + 6\,y = 0$.
7. $y = c_1e^{ax} + c_2e^{bx}$. *Ans.* $y'' - (a + b)y' + aby = 0$.
8. $xy = c + c^2x$. *Ans.* $x^4y'^2 = y'x + y$.
9. $y = (c_1 + x)e^{3x} + c_2e^x$. *Ans.* $y'' - 4\,y' + 3\,y = 2\,e^{3x}$.
10. $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$. *Ans.* $y''' - 6\,y'' + 11\,y' - 6\,y = 0$.
11. $r = c\sin\theta$. *Ans.* $r\cos\theta = r'\sin\theta$.
12. $r = e^{c\theta}$. *Ans.* $r\log r = r'\theta$.

13. Assuming the differential equation found in Ex. 1, indicate the values of $y'(= -x/y)$ at a large number of points (x, y) by short straight-line segments through each point in the correct direction. Continue doing this at points distributed over the plane until a set of curves is outlined. Are these the curves given in Ex. 1?

14. Proceed as in Ex. 13 for the equation $y' = y/x$. Do you recognize the set of curves? Can you *prove* that your guess is correct?

15. Draw a figure to illustrate the meaning of $y' = x^2$. Find y. Generalize the problem to the case $y' = f(x)$.

16. Find that curve of the set given in Ex. 1 which passes through $(1, 2)$. Find its slope (value of y') at that point. Do these three values of (x, y, y') satisfy the differential equation given as the answer in No. 1?

17. Proceed as in Ex. 16 for the equation of Ex. 2.

18. Proceed as in Ex. 16 for the first equation of Ex. 15.

19. Find the differential equation of all circles having their centers at the origin.

20. Find the differential equation of all parabolas with given latus rectum and axes coincident with the x-axis.

21. Find the differential equation of all parabolas with axes falling in the x-axis.

22. Find the differential equation of a system of confocal ellipses.

23. Find the differential equation of a system of confocal hyperbolas.

24. Find the differential equation of the curves in which the subtangent equals the abscissa of the point of contact of the tangent.

25. A point is moving at each instant in a direction whose slope equals the abscissa of the point. Find the differential equation of all the possible paths.

26. Write the differential equation of linear motion with constant acceleration; of linear motion whose acceleration varies as the square of the displacement. The same for angular motion of rotation.

27. A bullet is fired from a gun. Write the differential equations which govern its motion, air resistance being neglected. How must these equations be modified, if air resistance is assumed proportional to velocity?

181. General Statement. We shall now consider methods for solving differential equations. Since the most common properties of *curves* involve slope and curvature, and since in the theory of *motion* we deal constantly with speed and acceleration, *the differential equations of the first and second orders are of prime importance.*

Ordinary differential equations of the first order and first degree have the form

$$(1) \qquad M + N\frac{dy}{dx} = 0, \text{ or } M\,dx + N\,dy = 0,$$

where M and N are functions of x and y.

No general method is known for solving all such differential equations in terms of elementary functions. We proceed to give some standard methods of solution in special cases.

182. Type I. Separation of Variables. It may happen that M involves x only, and N involves y only. The variables are then said to be *separated* and the primitive is found by direct integration:

$$\int M dx + \int N dy = C,$$

C being an arbitrary constant.

Example 1. Find the curves having a constant subnormal equal to k. The differential equation is

$$\text{subnormal} = y \cdot \frac{dy}{dx} = k.$$

Separating the variables: $y\,dy = k\,dx$.

Integrating both sides: $\frac{1}{2} y^2 = kx + c,$

or $y^2 = 2\,kx + c',$

a family of parabolas. The constant c' is determined if the parabola is required to pass through some given point in the plane.

Check this result by eliminating c again by the methods of § 179.

Example 2. Given the relative rate of change (logarithmic derivative) of a function of x in terms of x, find the function: *i.e.* given

$$(dy/dx) \div y = \phi(x), \text{ to find } y = f(x).$$

The differential equation

$$\frac{dy}{dx} \div y = \phi(x)$$

is of the type mentioned above; separating variables and then integrating we find:

$$\frac{dy}{y} = \phi(x)dx, \text{ whence } \log y = \int \phi(x)\,dx + c, \text{ or } y = ke^{\int \phi(x)dx},$$

where $k = e^c$. If $\phi(x) = x$, for example, $y = ke^{x^2/2}$; if also the value of y is given for some value of x, say $y = 3$, when $x = 2$, we have $3 = ke^2$, whence $k = 3\,e^{-2}$ and $y = 3\,e^{x^2/2-2}$. Check this result.

183. Type II. Homogeneous Equations. When M and N are homogeneous * in x and y and of the same degree, the equation is said to be *homogeneous*. If we write the equation in the form

$$\frac{dy}{dx} = -\frac{M}{N},$$

and make the substitution

$$y = vx, \quad \frac{dy}{dx} = v + \frac{x\,dv}{dx},$$

we obtain a new equation in which the variables can be separated.

Example 1.

(1) $$(xy + y^2)\,dx + (xy - x^2)\,dy = 0,$$

or

(2) $$\frac{dy}{dx} = \frac{xy + y^2}{x^2 - xy}.$$

Substituting as above :

(3) $$v + x\frac{dv}{dx} = \frac{vx^2 + v^2x^2}{x^2 - vx^2} = \frac{v + v^2}{1 - v},$$

or $$x\frac{dv}{dx} = \frac{2\,v^2}{1 - v};$$

separating variables, $$\frac{1 - v}{2\,v^2}dv = \frac{dx}{x}.$$

Integrating : $$-\frac{1}{2\,v} - \frac{1}{2}\log v = \log x + c.$$

Replacing v by y/x,

$$-\frac{x}{2\,y} - \frac{1}{2}\log\frac{y}{x} = \log x + c,$$

or $$\log xy = -\frac{x}{y} - 2\,c;$$

hence

(4) $$xy = e^{-x/y - 2c},$$

or $$xy = ke^{-x/y},$$

where $$k = e^{-2c}.$$

* Polynomials are homogeneous in x and y when each term is of the same degree. In general, $f(x, y)$ is homogeneous if $f(kx, ky) = k^n f(x, y)$ for some one value of n and for all values of k.

Check: Differentiating both sides of (4) with respect to x, we find

$$y\,dx + x\,dy = ke^{-x/y}\left[-\frac{y\,dx - x\,dy}{y^2}\right]; \tag{5}$$

dividing the two sides of (5) by the corresponding sides of (4) respectively

$$[y\,dx + x\,dy] \div xy = -\frac{y\,dx - x\,dy}{y^2}; \tag{6}$$

show that (6) agrees with (1).

EXERCISES LXXIII. — SEPARATION OF VARIABLES

Solve the following exercises by separating the variables:

1. $x\,dy + y\,dx = 0$. *Ans.* $xy = c$.
2. $x\sqrt{1+y^2}\,dx - y\sqrt{1+x^2}\,dy = 0$. *Ans.* $\sqrt{1+x^2} = \sqrt{1+y^2} + c$.
3. $\sin\theta\,dr + r\cos\theta\,d\theta = 0$. *Ans.* $r\sin\theta = c$.
4. $x\sqrt{1+y}\,dx = y\sqrt{1+x}\,dy$.

Solve the following homogeneous equations:

5. $(x+y)\,dx + (x-y)\,dy = 0$. *Ans.* $x^2 + 2\,xy - y^2 = c$.
6. $(x^2+y^2)\,dx = 2\,xy\,dy$. *Ans.* $x^2 - y^2 = cx$.
7. $(3\,x^2 - y^2)\,dy = 2\,xy\,dx$. *Ans.* $x^2 - y^2 = cy^3$.
8. $(x^2 + 2\,xy - y^2)\,dx = (x^2 - 2\,xy - y^2)\,dy$. *Ans.* $x^2 + y^2 = c(x+y)$.

The following Ex. 9–18 are intended partially for practice in recognizing types:

9. $\sqrt{1-y^2}\,dx + \sqrt{1-x^2}\,dy = 0$. *Ans.* $\sin^{-1}x + \sin^{-1}y = c$.
10. $x^3\,dx + (3\,x^2y + 2\,y^3)\,dy = 0$. *Ans.* $x^2 + 2\,y^2 = c\sqrt{x^2+y^2}$.
11. $dy + y\sin x\,dx = \sin x\,dx$.
12. $r\,d\theta = \tan\theta\,dr$.
13. $(y-1)\,dx = (x+1)\,dy$.
14. $y\,dx + (x-y)\,dy = 0$.
15. $x(1+y^2)\,dx = y(1+x^2)\,dy$.
16. $(9\,x^2 + y^2)\,dx = 2\,xy\,dy$.
17. $\dfrac{dy}{dx} \div x = c$.
18. $\dfrac{dy}{dx} \div y = x$.

19. In Ex. 1 above, draw a figure to represent the direction of the integral curves at various points. Hence solve the equation geometrically.

20. A point moves so that the angle between the x-axis and the direction of the motion is always double the vectorial angle. Determine the possible paths. *Ans.* $\dfrac{xy}{x^2+y^2}=cx$; $c>0$.

21. Proceed as in Ex. 20 for a point moving so that its radius vector always makes equal angles with the direction of the motion and the x-axis. *Ans.* $r=c\sin\theta$.

22. The speed of a moving point varies jointly as the displacement and the sine of the time. Determine the displacement in terms of the time. *Ans.* $s=ce^{-k\cos t}$.

23. Find the value of y if its logarithmic derivative with respect to x is x^2.

184. Type III. Linear Equations. This name is applied to equations of the form

$$\frac{dy}{dx}+Py=Q, \tag{1}$$

where P and Q do not involve y, but may contain x. Its solution can be obtained by first finding a particular solution of the **reduced equation,**

$$\frac{dy^*}{dx}+Py^*=0, \tag{1*}$$

where y^* is a new quantity introduced for convenience in what follows; and where Q is replaced by zero. In (1*) the variables can be separated (see Ex. 2, § 182), and we get

$$y^*=e^{-\int P\,dx}$$

as a *particular* solution, the constant C of integration being given the particular value 0.

If we make the substitution

$$y=v\cdot y^*, \tag{2}$$

where v is a function of x to be determined, the equation (1) becomes

$$v\frac{dy^*}{dx}+y^*\frac{dv}{dx}+Pvy^*=Q,$$

or

$$v\left(\frac{dy^*}{dx}+Py^*\right)+y^*\frac{dv}{dx}=Q.$$

The first term vanishes by (1*) leaving

$$y^*\frac{dv}{dx}=Q,\quad\text{or}\quad dv=\frac{Q}{y^*}dx=[Qe^{\int P\,dx}]dx.$$

Hence

$$v=\int\frac{Q}{y^*}dx+c=\int[Qe^{\int P\,dx}]\,dx+c$$

and

(3) $$y=vy^*=e^{-\int P\,dx}\left\{\int[Qe^{\int P\,dx}]\,dx+c\right\}.$$

This equation expresses the solution of any linear equation. It should not be used as a formula; rather, the substitution (2) should be made in each example.

Example 1. Given

(1)′ $$\frac{dy}{dx}+3\,x^2y=x^5,$$

the reduced equation in the new letter $y^*=y/v$ is

(1*)′ $$\frac{dy^*}{dx}+3\,x^2y^*=0,\quad\text{whence}\quad y^*=e^{-x^3}.$$

Hence the substitution $y=v\cdot y^*$ becomes

(2)′ $$y=v\,e^{-x^3},\quad\text{whence}\quad\frac{dy}{dx}=e^{-x^3}\frac{dv}{dx}-3\,vx^2e^{-x^3},$$

and (1) takes the form

$$\left[e^{-x^3}\frac{dv}{dx}-3\,vx^2e^{-x^3}\right]+3\,x^2[ve^{-x^3}]=x^5.$$

This reduces, as we foresaw in general above, to the form

$$e^{-x^3}\frac{dv}{dx}=x^5,\quad\text{or}\quad\frac{dv}{dx}=x^5\,e^{x^3},$$

whence

$$v = \int x^5 e^{x^3}\,dx + c = \tfrac{1}{3}\left[x^3 e^{x^3} - e^{x^3}\right] + c,$$

or, returning by (2) to y:

$$(3)' \qquad y = ve^{-x^3} = \tfrac{1}{3}\left[x^3 - 1\right] + ce^{-x^3}.$$

Check: Differentiating both sides,

$$(4) \qquad \frac{dy}{dx} = x^2 - 3\,x^2 ce^{-x^3}\,;$$

eliminating c by multiplying $(3)'$ by $3\,x^2$ and adding to (4),

$$\frac{dy}{dx} + 3\,x^2 y = x^5.$$

The result $(3)'$ may also be obtained by direct substitution in (3) from $(1)'$. Sufficient practice in the direct solution, as in the preceding example, is strongly advised.

185. Extended Linear Equations. This name is often given to equations of the form

$$dy/dx + Py = Qy^n.$$

Putting $z = y^{1-n}$ reduces it to a linear equation in z.

Example 1. Given

$$\frac{dy}{dx} + \frac{y}{x} = xy^3. \quad \text{Put } z = y^{-2}.$$

Then $$\frac{dz}{dx} = -\,2\,y^{-3}\frac{dy}{dx}, \quad \text{or} \quad \frac{dy}{dx} = -\,(1/2)\,y^3\frac{dz}{dx}.$$

Thus $$-\,(1/2)\,y^3\frac{dz}{dx} + \frac{y}{x} = xy^3$$

and $$\frac{dz}{dx} - 2\,\frac{z}{x} = -\,2\,x.$$

Here $$P = -\frac{2}{x}, \quad \int P\,dx = -\,2\log x, \quad e^{\int P\,dx} = x^{-2}\,;$$

so that $$z = x^2\left(\int -\frac{2}{x}dx + c\right) = -\,2\,x^2\log x + cx^2 = y^{-2},$$

and finally $x^2y^2(c - 2\log x) = 1$. Check this result.

EXERCISES LXXIV. — LINEAR EQUATIONS

Solve the following linear equations and check each answer:

1. $\dfrac{dy}{dx} - xy = e^{x^2/2}$.

2. $\dfrac{dy}{dx} + 3x^2y = 3x^5$.

3. $\dfrac{dy}{dx} + y\cos x = \sin 2x$.

4. $x\dfrac{dy}{dx} + y = \log x$.

Solve the following extended linear equations, checking each answer:

5. $\dfrac{dy}{dx} + \dfrac{y}{x} = y^3$.

6. $\dfrac{dy}{dx} + y = xy^3$.

7. $\dfrac{dr}{d\theta} + r\theta = r^2\sin\theta$.

8. $xy^2\dfrac{dy}{dx} - y^3 = x^2$.

Solve the following equations, checking each answer:

9. $\cos^2 x\dfrac{dy}{dx} + y = \tan x$.

10. $r\dfrac{dr}{d\theta} = (1 + r^2)\sin\theta$.

11. $\dfrac{ds}{dt} = -s + t$. *Ans.* $s = ce^{-t} - 1 + t$.

12. $\dfrac{dy}{dx} + y = e^{-x}$. *Ans.* $ye^x = x + c$.

13. $dy - y\,dx = \sin x\,dx$.

14. $\sec\theta\,dr + (r - 1)\,d\theta = 0$.

15. $(x^2 + 1)\,dy = (xy + k)\,dx$.

16. $x\,dy + y\,dx = xy^2\log x\,dx$.

17. The equation of a variable electric current is

$$L\frac{di}{dt} + Ri = e,$$

where L and R are constants of the circuit, i is the current, and e the electromotive force of the circuit. Calculate i in terms of t, 1°, if e is constant; 2°, if $e = e_0 \sin\omega t$.

Ans. 2° $i = \dfrac{e_0}{\sqrt{R^2 + \omega^2L^2}}\sin(\omega t - \phi) + ce^{-Rt/L}$, $\phi = \arctan(\omega L/R)$.

186. Other Methods. Non-linear Equations. A variety of other methods are given in treatises on Differential Equations; some of these are indicated among the exercises which follow. Noteworthy among these are the possibility of making advantageous *substitutions;* and — what amounts to a special type of substitution — the possibility of writing the given equation in

the form of a *total differential*, $dz = 0$, where z is a known function of x and y, which leads to the general solution $z = \text{constant}$ (see Exs. 4–11, below).

Equations not linear in y' may often be solved. If the given equation can be solved for y', several values of y' may be found, each of which constitutes a differential equation: the general solution of the given equation means the totality of all of the solutions of all of these new equations (see Exs. 15–16, p. 362).

EXERCISES LXXV. — MISCELLANEOUS EXERCISES

1. Solve the equation $2\,y\,dy/dx + xy^2 = e^x$.

[Hint. Put $y^2 = v$; then $dv/dx = 2\,y\,dy/dx$, and the equation becomes $dv/dx + xv = e^x$, which can be solved by previous methods.]

2. Solve the equation $\cos y\,dy + \sin y \sec^2 x\,dx = \tan x\,dx$.

[Hint. Put $v = \sin y$, $u = \tan x$; then $dv = \cos y\,dy$, $du = \sec^2 x\,dx$; the equation becomes $dv + v\,du = u\,du/(1+u^2)$, which is linear.]

3. Solve the following equations, using the indicated substitutions:

(*a*) $y^2\,dy + (y^3 + x)\,dx = 0$. (Put $v = y^3$.)

(*b*) $s\,dt - t\,ds = 2\,s\,(t - s)\,dt$. (Put $s = tv$.)

(*c*) $x\,dy - y\,dx = (x^2 - y^2)\,dy$. (Put $y = vx$.)

(*d*) $u^2v^2\,(u\,dv + v\,du) = (v + v^2)\,dv$. Put $uv = x$, $v = y$.)

4. Solve the equation $(3\,x^2 + y)\,dx + (x + 3\,y^2)\,dy = 0$.

[Hint. If we put $z = x^3 + xy + y^3$, this equation reduces to $dz = 0$; for $dz = (\partial z/\partial x)\,dx + (\partial z/\partial y)\,dy$. But $dz = 0$ gives $z = \text{const.}$, hence $x^3 + xy + y^3 = c$ is the general solution. Such an equation as that given in this example is called an **exact differential equation**.]

5. Solve the equation $x\,dy - y\,dx = 0$.

Hint. This equation can be solved by previous methods; but it is easier to divide both sides by x^2 and notice that the resulting equation is $d(y/x) = 0$; hence the general solution is $y/x = c$. A factor which renders an equation exact ($1/x^2$ in this example) is called an **integrating factor**.

6. Solve the equation $(x^2 + 2\,xy^2)\,dx + (2\,x^2y + y^2)\,dy = 0$.

[Hint. Put $z = x^3/3 + x^2y^2 + y^3/3$.]

7. Solve the equation $(s + t \sin s)\, ds + (t - \cos s)\, dt = 0$.

[Hint. Arrange: $s\, ds + [t \sin s\, ds - \cos s\, dt] + t\, dt = 0$; integrate this knowing that the bracketed term is $-d(t \cos s)$.]

8. Solve the equation $x\, dy - (y - x)\, dx = 0$.

[Hint. Arrange: $[x\, dy - y\, dx] + x\, dx = 0$; divide by x^2, and compare Ex. 5.]

9. Show that $[f(x) + 2\, xy^2]\, dx + [2\, x^2 y + \phi(y)]\, dy = 0$ can always be solved by analogy to Ex. 6.

10. Show that $[f(x) + y]\, dx - x\, dy$ can always be solved by analogy to Ex. 5. Solve $(x^2 + y)\, dx - x\, dy = 0$. *Ans.* $x - y/x = c$.

11. Solve the equation $(r - \tan \theta)\, d\theta + (r \sec \theta + \tan \theta) dr = 0$.

[Hint. Multiply both sides by the *integrating factor* $\cos \theta$; $-\sin \theta\, d\theta + r\, dr + d(r \sin \theta) = 0$; integrate term by term.]

12. When a family of curves crosses those of another family everywhere at right angles, the curves of either family are called the **orthogonal trajectories** of those of the other family.

Find the orthogonal trajectories of the family of circles

$$x^2 + y^2 = r^2.$$

[Hint. If the differential equation of the first family be $dy/dx = f(x, y)$, then the differential equation of the orthogonal trajectories is $dx/dy = -f(x, y)$, for at any point of intersection (x, y) the slope of the curve of one system is the negative reciprocal of the slope of the curve of the other.

In this example the differential equation of the given family is $x\, dx + y\, dy = 0$. It is evident that the differential equation of the orthogonal family is obtained by replacing dy and dx by $-dx$ and dy, respectively; hence the desired equation is $x\, dy - y\, dx = 0$, whence the curves are $y = cx$, *i.e.* the family of all straight lines through the origin.]

13. Find the orthogonal trajectories of the exponential curves

$$y = e^x + k.$$

[Hint. The differential equation is $dy/dx = e^x$. The orthogonal family is defined by the equation $dy/dx = -e^{-x}$, whence the trajectories are $y = e^{-x} + c$. Draw the figure.]

14. Determine the orthogonal trajectories of the following families, and draw diagrams in illustration of each:

(*a*) $x + y = k$

(*b*) $xy = k$.

(c) $y^2 = 4\, k(x + k)$.

(*d*) $x^2 + y^2 = 2 \log x + c$.

(*e*) $2\, x^2 + y^2 = c^2$.

(*f*) $x^2 + y^2 = kx$.

15. Solve the equation $y'^2 - (x+y)y' + xy = 0$, where $y' = dy/dx$.

[HINT. Solving for y' we find $y' = x$ or $y' = y$. The general solutions of these *two* equations, which can easily be found by previous methods, can be written together in one equation by a principle of Analytic Geometry: $(2y - x^2 - c)(y - ce^x) = 0$.]

16. Solve each of the following equations:

(a) $y'^2 - 4y'x + 4x^2 - 1 = 0$. *Ans.* $(y - x^2)^2 - (x+c)^2 = 0$.

(b) $x^2y'^2 + 3xyy' + 2y'^2 = 0$. *Ans.* $(xy - c)(x^2y - c) = 0$.

(c) $y'(y' + y) = x(x+y)$. *Ans.* $(2y - x^2 - c)(y + x - 1 - ce^{-x}) = 0$.

(d) $y^2 + y'^2 = 1$. *Ans.* $y^2 = \cos^2(x + c)$.

(e) $y'^2 = 1 - x^2$. *Ans.* $2y = \pm(x\sqrt{1-x^2} + \text{arc} \sin x) + c$.

17. Solve the equation $2y = 4 + m^2$, where $m = dy/dx$.

[HINT. This may be solved as above, or by the following simpler process: Differentiate both sides with respect to x: $2m = 2m\,(dm/dx)$; solve this for m: $m = x + c$; substitute this value in the given equation: $2y = 4 + (x+c)^2$.]

18. Solve the equation $y = mx + m^2$.

[HINT. Proceeding as in Ex. 17, we find that the new equation in m is: $(x + 2m)(dm/dx) = 0$. If $dm/dx = 0$, $m = c$; substituting: $y = cx + c^2$.]

19. Find the envelope of the solutions of (18) and show that the envelope itself is a solution.

[HINT. See § 152, p. 298. The envelope is $y = -x^2/4$. Show this is a solution of the given equation (Ex. 18) by direct check. The envelope process demonstrates this also, for the values of (x, y, m) at any point of the envelope are the same as (x, y, m) on some curve of the set $y = cx + c^2$.]

20. In like manner, show that the envelope of any set of solutions of any differential equation is itself a solution. [This new solution is called **a singular solution.**]

21. Solve the following equations, find the envelope of the solutions in each case, and show that the envelope is a solution. [These are called **Clairaut** equations.]

(a) $y = mx - m^3$. *Ans.* $y = cx - c^3$; $y = \dfrac{2x}{9}\sqrt{3x}$.

(b) $y = mx - e^m$. *Ans.* $y = cx - e^c$; $y = x\log x - x$.

(c) $y = mx + f(m)$.

Ans. $y = cx + f(c)$; Eliminate c with $x + f'(c) = 0$.

PART II. ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

187. Special Types. We first consider some very special forms of equations of the second order that are most frequently used in the application of mathematics to physics, namely:

[I] $$\frac{d^2y}{dx^2} = \pm k^2 y \quad [k = \text{constant}].$$

[II] $$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = 0 \quad [A,\ B,\ C, \text{ constants}].$$

[III] $$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = F(x). \quad [A,\ B,\ C, \text{ constants.}]$$

These are all special forms of the general equation of the second order $\phi(x, y, dy/dx, d^2y/dx^2) = 0$.

[IV] We shall consider other special forms also, some of which include the above; namely, the cases that arise when one or more of the quantities x, y, dy/dx, are absent from the equation. (See § 191, p. 371.)

188. Type I. This type of equation arises in problems on *motion* in which the tangential acceleration d^2s/dt^2 is proportional to the distance passed over (see § 89, p. 156):

(1) $$\frac{d^2s}{dt^2} = \pm k^2 s,$$

a form which is equivalent to [I], written in the letters s and t. If we multiply both sides of this equation by the speed $v = ds/dt$ and then integrate with respect to t, we obtain

(2) $$\int \frac{ds}{dt}\frac{d^2s}{dt^2}\,dt = \int \pm k^2 s \frac{ds}{dt}\,dt;$$

but we know that

$$\int \frac{ds}{dt}\frac{d^2s}{dt^2}\,dt = \int v\frac{dv}{dt}\,dt = \int v\,dv = \frac{1}{2}v^2 + c = \frac{1}{2}\left(\frac{ds}{dt}\right)^2 + c,$$

and

$$\int \pm k^2 s \frac{ds}{dt}\,dt = \pm k^2 \int s\,ds = \pm \frac{k^2}{2} s^2 + c';$$

hence (2) becomes*

$$\frac{v^2}{2} = \frac{1}{2}\left(\frac{ds}{dt}\right)^2 = \pm \frac{k^2}{2}(s^2 + C_1). \tag{3}$$

Case 1. If the sign before k^2 is $+$, (3) becomes

$$v = \frac{ds}{dt} = k\sqrt{s^2 + C_1}, \tag{4}$$

whence $$\int \frac{ds}{\sqrt{s^2 + C_1}} = \int k\,dt + C_2,$$

$$\log\left(s + \sqrt{s^2 + C_1}\right) = kt + C_2; \tag{5}$$

or, solving for s,

$$s = Ae^{kt} + Be^{-kt}, \tag{6}$$

where $2A = e^{C_2}$ and $2B = -C_1 e^{-C_2}$ are two new arbitrary constants.

By means of the hyperbolic functions $\sinh u = (e^u - e^{-u})/2$ and $\cosh u = (e^u + e^{-u})/2$ this result may also be written in the form

$$s = a \sinh(kt) + b \cosh(kt), \tag{7}$$

where $$b + a = 2A \text{ and } b - a = 2B.$$

Case 2. If the sign before k^2 is $-$, C_1 must be negative also, or else v is imaginary; hence we set $C_1 = -a^2$ and write

$$v = \frac{ds}{dt} = k\sqrt{a^2 - s^2}, \tag{4_2}$$

or $$\int \frac{ds}{\sqrt{a^2 - s^2}} = \int k\,dt + C_2,$$

* This is often called the *energy integral*, for if we multiply through by the mass m, the expression $mv^2/2$ on the left is precisely the kinetic energy of the body.

whence

$$\sin^{-1}\left(\frac{s}{a}\right) = kt + C_2; \tag{5_2}$$

or solving for s:

$$s = a \sin (kt + C_2) = A \sin kt + B \cos kt, \tag{6_2}$$

where $A = a \cos C_2$ and $B = a \sin C_2$ are two new arbitrary constants.

Equation (6_2) is the characteristic equation of **simple harmonic motion**; the amplitude of the motion is a, the period is $2\pi/k$, and the phase is $-C_2/k$.

The differential equation (1) was first found in § 88, p. 155. We now see that the general simple harmonic motion (6_2) is the only possible motion in which the tangential acceleration is a negative constant times the distance from a fixed point; *i.e.* it is the only possible type of **natural vibration** under the assumptions of § 90, p. 157.

EXERCISES LXXVI. — TYPE I

1. Solve each of the following equations:

(a) $\dfrac{d^2s}{dt^2} = s.$ (b) $\dfrac{d^2s}{dt^2} = -s.$

(c) $\dfrac{d^2s}{dt^2} = 4s.$ (d) $\dfrac{d^2s}{dt^2} = -9s.$

2. Find the curves for which the flexion (d^2y/dx^2) is proportional to the height (y).

3. Determine the motion described by the equation of Ex. 1 (a) if the speed $v\ (= ds/dt)$ and the distance traversed s are both zero when $t = 0$.

4. Proceed as in Ex. 3 for Ex. 1 (b), and explain your result.

5. Write the solution of Ex. 1 (a) in terms of $\sinh t$ and $\cosh t$. Determine the arbitrary constants by the conditions of Ex. 3, and show that the final answer agrees precisely with that of Ex. 3.

6. Determine the motion described by the equation of Ex. 1 (b) if $v = 2$ and $s = 10$ when $t = 0$; if $v = 0$ and $s = 5$ when $t = 0$.

189. Type II. Homogeneous Linear Equations of the Second Order with Constant Coefficients. The form of this equation is

$$(1) \qquad A\frac{d^2y}{dx^2}+B\frac{dy}{dx}+Cy=0,$$

where A, B, C are constants.

The type just considered is a special case of this one. Following the indications of the results we obtained in § 188, it is natural to ask whether there are solutions of any one of the types we found in the special case:

Trial of e^{kx}. If we substitute $y=e^{kx}$ in (1) we obtain the equation:

$$(2) \qquad [Ak^2+Bk+C]e^{kx}=0.$$

The factor e^{kx} is never zero; *hence k must satisfy the quadratic equation*

$$(1\,{*}) \qquad Ak^2+Bk+C=0,$$

which is called the **auxiliary equation** to (1). If the roots of (1 *) are *real and distinct, i.e. if*

$$(3) \qquad D\equiv B^2-4\,AC>0,$$

then *these roots k_1 and k_2 are possible values for k*, and the general solution of (1) is

$$(4) \qquad y=C_1e^{k_1x}+C_2e^{k_2x},$$

since a trial is sufficient to convince one that the sum of two solutions of (1) is also a solution of (1); and that a constant times a solution is also a solution.

Trial of $y=e^{\kappa x}\cdot v$. If (3) is not satisfied, the substitution

$$(5) \qquad y=e^{\kappa x}\cdot v$$

changes (1) to the form

$$(6) \qquad A\frac{d^2v}{dx^2}+[2\,\kappa A+B]\frac{dv}{dx}+[A\kappa^2+B\kappa+C]v=0,$$

which becomes quite simple if we determine κ so that the term in dv/dx is zero:

(7) $$2\kappa A + B = 0, \quad \text{whence} \quad \kappa = -B/2A;$$

then (6) takes the form

(8) $$\frac{d^2v}{dx^2} = \frac{B^2 - 4AC}{4A^2}v = -\mathrm{K}^2 v,$$

where $\mathrm{K} = \sqrt{4AC - B^2}/(2A) = \sqrt{-D}/2A$ is *real* if

(9) $$D \equiv B^2 - 4AC \leqq 0,$$

which is the case we could not solve before.

If $D < 0$, the solutions of (8) are

(10) $$v = C_1 \sin(\mathrm{K}x) + C_2 \cos(\mathrm{K}x),$$

by (6_2), § 188, p. 365; hence the solutions of (1) are

(11) $$\boldsymbol{y = e^{\kappa x} \cdot v = e^{\kappa x}[C_1 \sin(\mathrm{K}x) + C_2 \cos(\mathrm{K}x)]},$$

where $\kappa = -B/(2A)$ and $\mathrm{K} = \sqrt{-D}/(2A)$; these values of κ and K are most readily found by solving (1 *) for k, *since the solutions of* (1 *) *are* $\boldsymbol{k} = (-\mathrm{B} \pm \sqrt{D})/(2A) = \boldsymbol{\kappa \pm \mathrm{K}\sqrt{-1}}$.

If $D = 0$, $\mathrm{K} = 0$ and the solutions of (8) are

(12) $$v = C_1 x + C_2;$$

hence the solutions of (1) are

(13) $$\boldsymbol{y = e^{\kappa x} \cdot v = e^{\kappa x}[C_1 x + C_2]},$$

where $\kappa = -B/(2A)$ is the solution of (1 *); since when $D = 0$, (1 *) has only one root $k = -B/(2A)$.

It follows that *the solutions of* (1) *are surely of one of the three forms* (4), (11), (13), *according as* $D = B^2 - 4AC$ *is* $+$, $-$, *or* 0; that is, *according as the roots of the* **auxiliary equation** (1 *) *are real and distinct, imaginary, or equal;* in résumé:

$D=B^2-4AC$	Character of Roots of (1*)	Values of Roots of (1*)	Solution of (1)
+	Real, unequal	k_1, k_2	(4)
−	Imaginary	$\kappa \pm \mathrm{K}\sqrt{-1}$	(11)
0	Equal	κ	(13)

Such solutions as (11) have been forecasted from the work of § 92, p. 162, where an equation (in the letters s and t) of precisely the type (11) was studied. Indeed, if y and x are replaced by s and t, and if κ is negative, (11) expresses precisely the most general form of **damped vibration**, studied in § 92.

Examples	1	2	3
Equation (1)	$3y''-4y'+y=0$	$3y''-4y'+\frac{4}{3}y=0$	$3y''-4y'+2y=0$
Auxiliary equation (1*)	$3k^2-4k+1=0$	$3k^2-4k+\frac{4}{3}=0$	$3k^2-4k+2=0$
Roots of (1*)	1, 1/3	2/3, 2/3	$\frac{1}{3}(2\pm\sqrt{-2})$
Solution of (1)	$y=c_1e^x+c_2e^{x/3}$	$y=e^{2x/3}(c_1+c_2x)$	$y=e^{2x/3}(c_1\cos\frac{\sqrt{2}}{3}x+c_2\sin\frac{\sqrt{2}}{3}x)$

EXERCISES LXXVII. — LINEAR HOMOGENEOUS. TYPE II

1. $y''-4y'+3y=0.$
2. $y''+3y'+2y=0.$
3. $5y''-4y'+y=0.$
4. $9y''+12y'+4y=0.$
5. $y''-2y'+y=0.$
6. $y''+y'+y=0.$
7. $y''-2y'+3y=0.$
8. $3y''+5y'+2y=0.$
9. $y''-9y'+14y=0.$
10. $2y''-3y'+y=0.$
11. $6y''-13y'+6y=0.$
12. $y''-3y'=0.$
13. $y''-4y=0.$
14. $y''+9y=0.$
15. $y''+ky'=0.$
16. $y''\pm ky=0.$

17. If a particle is acted on by a force that varies as the distance and by a resistance proportional to its speed, the differential equation of its motion is

$$d^2x/dt^2 + b\,dx/dt + cx = 0,$$

where $c > 0$ if the force attracts, and $c < 0$ if the force repels. Solve the equation in each case.

18. If in Ex. 17, $b = c = 1$, and the particle starts from rest at a distance 1, determine its distance and speed at any time t. Is the motion oscillatory? If so, what is the period? Solve when the initial speed is v_0.

19. If in Ex. 17, $b = 1$ and $c = -1$, discuss the motion as in Ex. 18.

190. Type III. Non-homogeneous Equations. This type is of the form:

(1) $$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = F(x),$$

where A, B, C, are constants, and $F(x)$ is a function of x only. We proceed to show that this form can be solved in a manner exactly analogous to § 184, p. 356; first write down the **reduced equation** in the new letter y^*:

(1*) $$A\frac{d^2y^*}{dx^2} + B\frac{dy^*}{dx} + Cy^* = 0,$$

and solve (1*) by the method of § 189. Let $y^* = \phi(x)$ be *any one* particular solution of (1*) (the simpler, the better, except that $y^* = 0$ is excluded). Then the substitution

(2) $$y = \phi(x) \cdot u$$

transforms (1) into

(3) $$\{A\phi''(x) + B\phi'(x) + C\phi(x)\}u + \{2A\phi'(x) + B\phi(x)\}\frac{du}{dx} + A\phi(x)\frac{d^2u}{dx^2} = F(x);$$

but, since $\phi(x)$ satisfies (1*), the first term of (3) is zero; and if we now set $du/dx = v$ temporarily, this equation can be written as the linear equation:

(4) $$\frac{dv}{dx} + \left\{\frac{2\,A\phi'(x) + B\phi(x)}{A\phi(x)}\right\} v = \frac{F(x)}{A\phi(x)},$$

which is precisely of the form solved in § 184. Comparing (4) with (1), § 184, we have

(5) $$P = \frac{2\,A\phi'(x) + B\phi(x)}{A\phi(x)},\quad Q = F(x)/[A\phi(x)].$$

Having found v by § 184, we have

$$u = \int v\,dx + c_2,\quad y = u\phi(x) = \phi(x)\left[\int v\,dx + c_2\right],$$

which is the required solution of (1).

Example 1. Given the equation

(1) $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2\,y = \sin x,$$

we write the reduced equation

(1*) $$\frac{d^2y^*}{dx^2} + 3\frac{dy^*}{dx} + 2\,y^* = 0;$$

this is easily solved by the method of § 189; the simplest particular solution is $y = e^{-x}$. Substituting $\phi(x) = e^{-x}$ in the general work above, we find

$$P = \frac{2\,A\phi'(x) + B\phi(x)}{A\phi(x)} = 1 \text{ and } Q = \frac{F(x)}{A\phi(x)} = e^x \sin x;$$

hence $e^{\int P\,dx} = e^x$, and

$$v = e^{-x}\left[\int e^{2x}\sin x\,dx + C_1\right] = \frac{1}{5}e^x(2\sin x - \cos x) + C_1e^{-x},$$

$$u = \int v\,dx + C_2 = \frac{1}{10}\,e^x(\sin x - 3\cos x) - C_1e^{-x} + C_2,$$

$$y = u\cdot\phi(x) = \frac{1}{10}(\sin x - 3\cos x) - C_1e^{-2x} + C_2e^{-x}.$$

EXERCISES LXXVIII. — NON-HOMOGENEOUS TYPE

1. $y'' - 3y' + 2y = \cos x$.

   Ans. $y = \frac{1}{10}(\cos x - 3\sin x) + c_1 e^x + c_2 e^{2x}$.

2. $y'' - 4y' + 2y = x$.

   Ans. $y = \frac{1}{2}(x+2) + c_1 e^{(2+\sqrt{2})x} + c_2 e^{(2-\sqrt{2})x}$.

3. $y'' + 3y' + 2y = e^x$.

   Ans. $y = e^x/6 - c_1 e^{-2x} + c_2 e^{-x}$.

4. $y'' - 2y' + y = x$. *Ans.* $y = x + 2 + e^x(c_1 + c_2 x)$.

5. $y'' + y = \sin x$. *Ans.* $y = -\frac{1}{2}x\cos x + c_1 \sin x + c_2 \cos x$.

6. $y'' - y' - 2y = \sin x$.

   Ans. $y = \frac{1}{10}(\cos x - 3\sin x) + c_1 e^{-x} + c_2 e^{2x}$.

7. $y'' + 4y = x^2 + \cos x$.

   Ans. $y = \frac{1}{8}(2x^2 - 1) + \frac{1}{3}\cos x + c_1 \cos 2x + c_2 \sin 2x$.

8. $y'' - 2y' = e^{2x} + 1$. *Ans.* $y = \frac{1}{2}x(e^{2x} - 1) + c_1 + c_2 e^{2x}$.

9. $y'' - 4y' + 3y = 2e^{3x}$. *Ans.* $y = xe^{3x} + c_1 e^x + c_2 e^{3x}$.

10. If a particle moves under the action of a periodic force through a medium resisting as the speed, the equation of motion is

$$d^2s/dt^2 + A\,ds/dt = B\sin Ct.$$

Express s and the speed in terms of t. If $A = B = C = 1$, what is the distance passed over and the speed after 5 seconds, the particle starting from rest?

191. Type IV. One of the quantities x, y, y' absent.

Type IV$_a$: $\phi(y'') = 0$. Solve for y'', to obtain a solution, say $y'' = a$. Then integrate twice. The general solution for each value of y'' is of the form $y = \frac{1}{2}ax^2 + c_1 x + c_2$.

In problems of *motion*, this type is equivalent to the statement that $\phi(j_T) = 0$, where $j_T = d^2s/dt^2 = dv/dt$. Hence j_T may have any one of the several *constant* values which satisfy $\phi(j_T) = 0$; but if $j_T = k$, $s = kt^2/2 + c_1 t + c_2$ (see Ex. 4, p. 73).

Type IV$_b$: y missing. $\phi(x, y', y'') = 0$. The substitution $m = y' = dy/dx$, $dm/dx = d^2y/dx^2 = y''$, reduces the given equation to an equation of the *first order* in m, x, dm/dx. Solving, if possible, one gets a relation of the form $f(m, x, c) = 0$. This

is again an equation of the first order in x and y, and may be integrated by methods given in Part I, §§ 182–186.

The interpretation in motion problems is particularly vivid and beautiful. Thus $v = ds/dt$ and $j_T = dv/dt = d^2s/dt^2$; hence any equation in j_T, v, t, *with s absent*, is a differential equation of the first order in v. Solving this, we get an equation in v and t; since $v = ds/dt$, this new equation is of the first order in s and t.

Example 1. $$1 + x + x^2\frac{d^2y}{dx^2} = 0.$$

Setting $dy/dx = m$, $$1 + x + x^2\frac{dm}{dx} = 0.$$

Separating variables, $$-\,dm = \frac{1+x}{x^2}dx.$$

Integrating, $$-\,m = -\frac{1}{x} + \log x + c_1.$$

Integrating again, $$y = \log x - x\log x + (1 - c_1)x - c_2.$$

Interpret this as a problem in motion, with s and t in place of y and x, and $j_T = dv/dt = d^2s/dt^2$.

Example 2. In a certain motion the space passed over s, the speed v, and the acceleration j_T are connected with the time by the relation $1 + v^2 - j_T = 0$; find s in terms of t.

Placing $j_T = dv/dt$, the equation

$$1 + v^2 - \frac{dv}{dt} = 0$$

is of the first order. The variables can be separated, and the integral is

$$\tan^{-1} v = t + c_1 \text{ or } v = \tan(t + c_1),$$

which is itself a differential equation of the first order if we replace v by ds/dt. Integrating this new equation:

$$\int ds = \int \tan(t + c_1)dt + c_2, \text{ or } s = -\log\cos(t + c_1) + c_2.$$

In such a motion problem we usually know the values of v and s for some value of t. If $v = 0$ and $s = 10$ when $t = 0$, for example, c_1 must be zero (or else a multiple of π) and c_2 must be 10; hence $s = -\log\cos t + 10$.

Example 3. $$1 + x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 0 = 1 + xm + x^2\frac{dm}{dx}.$$

This can be written $\quad dm/dx + m/x = -1/x^2$,

which is linear in m and x, the solution being

$$m = -\frac{1}{x}\log x + \frac{c_1}{x}.$$

The second integration gives

$$y = -\tfrac{1}{2}[\log x]^2 + c_1 \log x + c_2.$$

Interpret this as a motion problem, and determine c_1 and c_2 to make $y = 10$ and $m = 3$ when $x = 1$.

Type IV$_c$: x missing. $\phi(y, y', y'') = 0$. The substitution $m = y'$ gives

$$y' = m, \qquad y'' = \frac{dy'}{dx} = \frac{dy'}{dy}\cdot\frac{dy}{dx} = \frac{dm}{dy}\cdot m;$$

and the transformed equation is an equation of the first order in y and m. We solve this and then restore y' in place of m, whereupon we have left to solve another equation (in x and y) of the first order.

This is precisely the way in which we solved Type I, § 188, Type I being only an *important* special case of Type IV$_c$.

Example 1. If the acceleration j_T is given in terms of the distance passed over (compare § 188), we have

$$j_T = \frac{d^2s}{dt^2} = \phi(s), \quad \text{or} \quad \frac{dv}{dt} = \phi(s).$$

This is transformed by the relation

$$j_T = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v,$$

(which is itself *a most valuable formula*) into

$$\frac{dv}{ds}v = \phi(s)$$

in which the variables can be separated; integration gives

$$\frac{1}{2}v^2 = \int \phi(s)ds + c,$$

which is called the **energy integral** (see footnote, p. 364).

The work cannot be carried further than this without knowing an exact expression for $\phi(s)$. When $\phi(s)$ is given, we proceed as in § 188, replacing v by ds/dt and integrating the new equation:

$$\int \frac{ds}{\sqrt{2\int \phi(s)ds + 2c}} = t + k.$$

Unfortunately the indicated integrations are difficult in many cases; often they can be performed by means of a table of integrals. One case in which the integrations are comparatively easy is that already done in § 188.

EXERCISES LXXIX. — TYPE IV

1. $y''^2 - 4x^2 = 0.$ *Ans.* $y = \pm 1/3\, x^3 + c_1 x + c_2.$

2. $y'' = \sqrt{1 + y'^2}.$ *Ans.* $2y = c_1 e^x + e^{-x}/c_1 + c_2.$

3. $xy'' + y' = x^2.$ *Ans.* $y = x^3/9 + c_1 \log x + c_2.$

4. $s\, d^2s/dt^2 + (ds/dt)^2 = 1.$ *Ans.* $s^2 = t^2 + c_1 t + c_2.$

5. $\dfrac{d^2s}{dt^2} = \dfrac{1}{\sqrt{s}}.$

6. $\dfrac{d^2y}{dx^2} = \pm k^2 y.$

7. $\dfrac{d^2y}{dx^2} = e^{2x}.$

8. $\dfrac{d^2y}{dx^2} = e^{2y}.$

9. $\dfrac{d^2y}{dx^2} = x^2 \cos x.$

10. $\dfrac{d^3y}{dx^3} = x + 3 \sin x.$

11. $\dfrac{d^4y}{dx^4} = e^x - \cos 2x.$

12. $\dfrac{d^2y}{dx^2} = \left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}.$

13. Show that Ex. 12 is equivalent to the problem, to find a curve whose radius of curvature is unity.

14. The flexion (d^2y/dx^2) of a beam rigidly embedded at one end, and loaded at the other end, which is unsupported, is $k(l - x)$, where k is a constant and l is the length of the beam. Find y, and determine the constants of integration from the fact that $y = 0$ and $dy/dx = 0$ at the embedded end, where $x = 0$.

15. Find the form of a uniformly loaded beam of length l, embedded at one end only, if the flexion is proportional to $l^2 - 2lx + x^2$, where $x = 0$ at the embedded end.

16. Find the form of a uniformly loaded beam of length l, freely supported at both ends, if the flexion is proportional to $l^2 - 4x^2$ in each half, where x is measured horizontally from the center of the beam.

PART III. GENERALIZATIONS

192. Ordinary Equations of Higher Order. An equation whose order is *greater than two* is called **an equation of higher order**; the reason for this is the comparative rarity in applications of equations above the second order. There seems to be a natural line of division between order two and higher orders, which is analogous to the natural demarkation between space of three dimensions and space of higher dimensions.

We shall state briefly the generalizations to equations of higher order, however, since they do occur in a few problems, and since it is interesting to know that *practically the same rules apply in certain types* for higher orders as those we found for order two.

193. Linear Homogeneous Type. The work of § 189 can be generalized to any **linear homogeneous equation with constant coefficients**:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = 0. \tag{1}$$

Thus if we set $y = e^{kx}$, as in § 189, we find

$$k^n + a_1 k^{n-1} + \cdots + a_{n-1}k + a_n = 0, \tag{1*}$$

again called the **auxiliary** equation. Corresponding to any real root k_1 there is therefore a solution $e^{k_1 x}$; *if all the roots are real and distinct, the general solution of* (1) *is*

$$y = C_1 e^{k_1 x} + C_2 e^{k_2 x} + \cdots + C_n e^{k_n x}, \tag{2}$$

where $k_1, k_2, \cdots, k_n$ are the roots of (1). Curiously enough, the chief difficulty is not in any operation of the Calculus; rather it is in solving the algebraic equation (1*).

It is easy to show by extensions of the methods of § 189

that *any pair of imaginary roots of* (1*), $k = \kappa \pm \mathrm{K}\sqrt{-1}$ *corresponds to a solution of the form* †

$$(3) \qquad y = e^{\kappa x}[C' \sin(Kx) + C'' \cos(Kx)],$$

which then takes the place of two of the terms of (2).

Finally, *if a root* $k = \kappa$ of (1*) *occurs more than once, i.e.* if the left-hand side of (1*) has a factor $(k - \kappa)^p$, *the corresponding solution obtained as above should be* **multiplied** *by the polynomial*

$$(4) \qquad B_0 + B_1x + B_2x^2 + \cdots + B_{p-1}x^{p-1},$$

where p is the order of multiplicity of the root (*i.e.* the exponent of $(k - \kappa)^p$), and where the B's are arbitrary constants which replace those lost from (2) by the condensation of several terms into one.

The proof is most easily effected by making the substitution $y = e^{\kappa x} \cdot u$, whereupon the transformed differential equation contains no derivative below d^pu/dx^p; hence $u =$ the polynomial (4) is a solution of the new equation, and $y = e^{\kappa x}$ times the polynomial (4) is a solution of (1). This work may be carried out by the student in any example below in which (1*) has multiple roots.‡

† This fact is often made plausible by the use of the equations

$$e^{u\sqrt{-1}} = \cos u + \sqrt{-1}\sin u, \quad e^{-u\sqrt{-1}} = \cos u - \sqrt{-1}\sin u;$$

these equations can be derived formally by using the Taylor series for e^v, $\cos u$, $\sin u$, with $v = u\sqrt{-1}$, but they remain only plausible until after a study of the theory of imaginary numbers. The solutions $e^{\kappa \pm K\sqrt{-1}}$ are indicated formally by (2); hence it is plausible that (3) is correct.

A more direct process which avoids any uncertainty concerning imaginaries is almost as easy. For the substitution $y = e^{\kappa x} \cdot u$ (see § 189) gives a new equation in u and x which, together with its auxiliary, has coefficients of the form $(d^nA(k)/dk^n) \div n!$, where $A(k)$ represents the left-hand side of (1*). Now $K\sqrt{-1}$ is a solution of the new auxiliary by development of $A(k)$ in powers of $(k - \kappa)$; hence $u = \sin(Kx)$ and $u = \cos(Kx)$ are solutions of the new differential equation, as a comparison of coefficients demonstrates. This process constitutes a rigorous proof of (3).

‡ To avoid using imaginary powers of e, if that is desired, substitute $y = e^{\kappa x}[\cos(Kx) + \sqrt{-1}\sin(Kx)]u$, when the multiple root is imaginary, $k = \kappa + K\sqrt{-1}$.

These extensions of § 189 should be verified by the student by a direct check in each exercise.

EXAMPLE	1	2	3
(1)	$y''' - y' = 0$	$y^{\text{iv}} + 6\,y''' + 12\,y'' + 8\,y' = 0$	$y''' + 8\,y = 0$
(1*)	$k^3 - k = 0$	$k^4 + 6\,k^3 + 12\,k^2 + 8\,k = 0$	$k^3 + 8 = 0$
$k =$	$0,\ 1,\ -1$	$0,\ -2,\ -2,\ -2$	$-2,\ 1 \pm \sqrt{3}\sqrt{-1}$
y	$c_1 + c_2e^x + c_3e^{-x}$	$c_1 + e^{-2x}(c_2 + c_3x + c_4x^2)$	$c_1e^{-2x} + e^x(c_2 \cos\sqrt{3}\,x$ $+ c_3 \sin\sqrt{3}\,x)$

194. Non-homogeneous Type. The non-homogeneous type

(1) $$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n = F(x)$$

cannot be solved in general by an extension of § 190. But in the majority of cases which actually arise in practice,* a sufficient method consists in *differentiating* both sides of (1) repeatedly until an elimination of the *right*-hand sides becomes possible. The new equation will be of higher order still:

(2) $$\frac{d^my}{dx^m} + A_1\frac{d^{m-1}y}{dx^{m-1}} + \cdots + A_{m-1}\frac{dy}{dx} + A_m = 0,$$

but its right-hand side is zero. Solve this equation by § 193, and then substitute the result in (1) for trial; of course there will be too many arbitrary constants; the superfluous ones are determined by comparison of coefficients, as in the examples below.

Example 1. $y''' + y' = \sin x$.

Differentiating both sides twice and adding the result to the given equation :

$$y^{\text{v}} + 2\,y''' + y' = 0.$$

* For more general methods, see any work on Differential Equations ; *e.g.* Forsyth, *Differential Equations.*

The auxiliary equation $k^5 + 2k^3 + k = 0$ has the roots $k=0$, $k = \pm\sqrt{-1}$ (twice). Hence we first write as a trial solution y_t the solution of the new equation: $y_t = c_1 + (c_2 + c_3x)\cos x + (c_4 + c_5x)\sin x$; substituting this in the given equation, we find $-2c_3\cos x - 2c_5\sin x = \sin x$, whence $c_3 = 0$ and $c_5 = -1/2$; substituting these values in the trial solution y_t gives the general solution of the given equation:

$$y = c_1 + c_2\cos x + (c_4 - x/2)\sin x.$$

EXERCISES LXXX. — LINEAR EQUATIONS OF HIGHER ORDER

1. $y''' - 3y'' = 0$. *Ans.* $y = c_1 + c_2x + c_3e^{3x}$.

2. $y''' - y'' - 4y' + 4y = 0$. *Ans.* $y = c_1e^x + c_2e^{2x} + c_3e^{-2x}$.

3. $y^{\text{iv}} - 16y = 0$. *Ans.* $y = c_1e^{2x} + c_2e^{-2x} + c_3\cos 2x + c_4\sin 2x$.

4. $y^{\text{iv}} - 6y'' + 9 = 0$. *Ans.* $y = e^{x\sqrt{3}}(c_1 + c_2x) + e^{-x\sqrt{3}}(c_3 + c_4x)$.

5. $y^{\text{v}} + 6y''' + 9y' = 0$.
Ans. $y = c_1 + (c_2 + c_3x)\cos\sqrt{3}\,x + (c_4 + c_5x)\sin\sqrt{3}\,x$.

6. $y^{\text{vi}} - 16y''' + 64y = 0$, $k = 2,\ 2,\ -1 \pm i\sqrt{3},\ -1 \pm i\sqrt{3}$.
Ans. $y = e^{2x}(c_1 + c_2x) + e^{-x}[(c_3 + c_4x)\cos\sqrt{3}\,x + (c_5 + c_6x)\sin\sqrt{3}\,x]$.

7. $y'' - 5y' + 4y = e^{2x}$. *Ans.* $y = c_1e^x - (1/2)e^{2x} + c_2e^{4x}$.

8. $3y'' + 4y' + y = \sin x$.

9. $y''' - 3y'' + 2y' = x$.

10. $y''' - y'' - 4y' + 4y = e^x$.

11. $y^{\text{iv}} - 5y'' + 4y = e^{2x}$.

12. Solve the equation $y''' + y' = 0$ by first setting $y' = p$.

13. Solve the following equations by setting $y' = p$ or else $y'' = q$.

(*a*) $3y''' - 4y'' + y' = 0$.
(*b*) $y''' + y'' + y' = 0$.
(*c*) $y''' + y' = \sin x$.
(*d*) $y''' + 3y'' + 2y' = e^x$.
(*e*) $y^{\text{iv}} - y'' = 0$.
(*f*) $y^{\text{iv}} + y'' = e^x$.

14. The following equations, though not linear, may be solved by first setting $y' = p$ or $y'' = q$ or $y''' = r$.

(*a*) $y' = y'' + \sqrt{1 + y''^2}$.
(*b*) $y'' + y'''x = (y'')^2x^4$.
(*c*) $1 + x + x^2y''' = 0$.
(*d*) $xy^{\text{iv}} + y''' = x^2$.

15. Solve the equation $x^2y'' + xy' - y = \log x$.

[Hint. Put $x = e^z$; then

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz};\quad \frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{1}{x}\frac{dy}{dz}\right)\cdot\frac{dz}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right);$$

so that the transformed equation is

$$\frac{d^2y}{dz^2} - y = z,\ \text{whence}\ y = c_1e^z + c_2e^{-z} - z = c_1x + c_2x^{-1} - \log x.]$$

16. Solve the equations,

(*a*) $x^2y'' - xy' - 3y = 0$. (*b*) $xy'' - y' = \log x$

(*c*) $(x+1)^2 y'' - 4(x+1)y' + 6y = x$, $(x + 1 = e^z)$.

(*d*) $(a+bx)^2y'' + (a+bx)y' - y = \log(a+bx)$, $(a + bx = e^z)$.

(*e*) $x^3y''' - 6y = 1 + x$.

195. Systems of Differential Equations. Let us finally consider systems of two equations, and let us suppose the equations to be linear in the derivatives, that is, to involve only the first powers of these derivatives.

196. Linear System of the First Order. Let the equations be

$$y' = ax + by + cz + d, \tag{1}$$

$$z' = a_1x + b_1y + c_1z + d_1, \tag{2}$$

where the coefficients are constant. We wish to determine y and z as functions of x.

Differentiating (1) with respect to x gives

$$y'' = a + by' + cz'; \tag{3}$$

then the elimination of z and z' between the three equations (1), (2), (3), gives a differential equation of the second order in y, which should be solved for y.

197. $dx/P = dy/Q = dz/R$. Here P, Q, and R are functions of x, y, z. Let λ, μ, ν be any multipliers, either constants or functions of x, y, z. Then, by the laws of algebra,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{\lambda dx + \mu dy + \nu dz}{\lambda P + \mu Q + \nu R}. \tag{1}$$

Suppose that we can select from these ratios (or from these together with others obtainable from them by giving suitable values to λ, μ, ν) two equal ratios free from z, *i.e.* containing only x and y. Such an equation is an ordinary differential equation of the first order in x and y. Solving it, we obtain

$$f(x, y, c_1) = 0. \tag{2}$$

Suppose that a second pair of ratios can be found, free from

another of the variables, say y. The result is an equation of the first order in x and z. Let its solution be

(3) $$F(x, z, c_2) = 0.$$

Then (2) and (3) form the complete solution of the system. Conversely, differentiating (2) and (3) with respect to x, eliminating c_1 and c_2, and solving for $dx:dy:dz$, we find a system like (1). In selecting the second pair of ratios, the result (2) of the first integration may be utilized to eliminate the variable whose absence is desired.

Example 1. $$dx/x^2 = dy/xy = dz/z^2.$$

The first two ratios give $dx/x = dy/y$, whence $y = c_1x$. Putting this value of y in $dy/xy = dz/z^2$ gives $dy/(c_1y^2) = dz/z^2$, so that

$$\frac{1}{c_1y} = \frac{1}{z} + c_2,$$

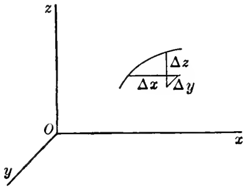


Fig. 76

or, $z = c_1y + c_1c_2yz = x + c_2xz$. Hence the solutions are given by the two equations $y = c_1x$, $z = x + c_2xz$.

Interpreted geometrically, the solutions represent a family of planes and a family of hyperboloids. These are the **integral surfaces** of the differential equation. Each plane cuts each hyperboloid in a space curve, forming a doubly infinite system of curves, the **integral curves** of the differential equation. The system may be written $dx : dy : dz = x^2 : xy : y^2$. But the direction cosines of the tangent to a space curve are proportional to dx, dy, dz. Thus the given equations define at each point a direction whose cosines are proportional to x^2, xy, y^2. Our solution is a system of curves having at each point the proper direction. What curve of the above system goes through (4, 2, 3)? What are the angles which the tangent to the curve at this point makes with the coördinate axes?

Example 2. $$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}.$$

Let $\lambda = \mu = \nu = 1$. Then each of the above fractions equals

$$\frac{dx + dy + dz}{0}.$$

But since the given ratios are in general finite, this gives

$$dx + dy + dz = 0, \quad \text{whence} \quad x + y + z = c_1.$$

Again, let $\lambda = x$, $\mu = y$, $\nu = z$. This gives

$$x\,dx + y\,dy + z\,dz = 0, \quad \text{whence} \quad x^2 + y^2 + z^2 = c_2.$$

Thus the integral surfaces are planes and spheres, and the integral curves are the circles in which they intersect.

In this example the multipliers λ, μ, ν have been chosen so as to get exact differentials.

Example 3. $$\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{z}.$$

The first two ratios are free from z and give

$$\arctan(y/x) = \log\left[c_1x^2/\sqrt{x^2+y^2}\right].$$

Using the multipliers $\lambda = x$, $\mu = y$, $\nu = 0$, and equating the ratio thus obtained to the last of the given ratios, we find

$$\frac{x\,dx + y\,dy}{x^2+y^2} = \frac{dz}{z}, \quad \text{whence} \quad x^2 + y^2 = c^2z^2.$$

EXERCISES LXXXI. — SYSTEMS OF EQUATIONS

1. $x\,dx/y^2 = y\,dy/x^2 = dz/z$. *Ans.* $x^4 - y^4 = c_1$; $z^2 = c_2(x^2 + y^2)$.

2. $dx/x = dy/y = -dz/z$. *Ans.* $yz = c_1$; $y = c_2x$.

3. $dx/yz = dy/xz = dz/(x + y)$.
Ans. $z^2 = 2(x + y) + c_1$; $x^2 - y^2 = c_2$.

4. $dx/(y + z) = dy/(x + z) = dz/(x + y)$.
Ans. $(x - y) = c_1(x - z) = c_2(y - z)$.

5. $dx/(x^2 + y^2) = dy/(2\,xy) = dz/(xz + yz)$.
Ans. $2\,y = c_1(x^2 - y^2)$; $x + y = c_2z$.

6. $\dfrac{dy}{dx} = \dfrac{2\,xy}{x^2 - y^2 - z^2}$; $\dfrac{dz}{dx} = \dfrac{2\,xz}{x^2 - y^2 - z^2}$.
Ans. $y = c_1z = c_2(x^2 + y^2 + z^2)$.

7. $\dfrac{dy}{dx} = \dfrac{z - 3\,x}{3\,y - 2\,z}$; $\dfrac{dz}{dx} = \dfrac{2\,x - y}{3\,y - 2\,z}$.
Ans. $x + 2\,y + 3\,z = c_1$; $x^2 + y^2 + z^2 = c_2$.

8. $dx = -ky\,dt$; $dy = kx\,dt$.
Ans. $x = A\cos kt + B\sin kt$; $y = A\sin kt - B\cos kt$.

9. $dx/dt = 3\,x - y$; $dy/dt = x + y$.
Ans. $x = (A + Bt)e^{2t}$; $y = (A - B + Bt)e^{2t}$.

10. Determine the curves in which the direction cosines of the tangent are respectively proportional to the coördinates of the point of contact; to the squares of those coördinates.

11. A particle moves in a plane so that the sum of the axial components of the speed always equals the sum of the coördinates of the particle, while the difference of the components is a constant k. Determine the possible paths. *Ans.* $x + y = c_1e^t$; $x - y = kt + c_2$.

12. If the particle in Exercise 11 is at (1, 1) when $t = 0$, where is it when $t = 5$? Approximately how far has it traveled?

198. Partial Differential Equations. While a general treatment of differential equations which involve *partial* derivatives is beyond the scope of this book, a few examples that can be solved without special theory will illustrate the nature of such equations and their solutions.

Example 1. Solve the equation $\partial z/\partial x = 0$, where z is some function of x and y.

Since $\partial z/\partial x = 0$, we have $z =$ "const." — *in so far as x is concerned.* But during a partial differentiation with respect to x, y acts like a constant; hence any arbitrary function of y, $A(y)$, may be put in place of the constant of integration, and the general solution is: $z = A(y)$.

Example 2. Solve the equation $\partial z/\partial x = 2x + y$.

Since y may be thought of as a constant during the integration with respect to x, we may integrate at once term by term, thinking of y as a constant: $z = x^2 + xy + A(y)$, where the arbitrary function of y, $A(y)$, takes the place of the usual constant of integration, as in Ex. 1.

Example 3. Solve the equation $\partial^2 z/\partial x\, \partial y = 2x + y$.

Integrating first with respect to x, we find, by a repetition of the work of Ex. 2, $\partial z/\partial y = x^2 + xy + A(y)$. Integrating again, this time with respect to y, thinking of x as constant, we find

$$z = x^2y + \frac{xy^2}{2} + \int A(y)\, dy + B(x),$$

where $B(x)$ is any arbitrary function of x, — a constant with respect to y. Since $\int A(y)dy$ is itself completely arbitrary, the general solution may be written

$$z = x^2y + \frac{xy^2}{2} + f(x) + \phi(y),$$

where $f(x)$ and $\phi(y)$ are arbitrary functions of x and y.

199. Relation to Systems of Ordinary Equations. It is shown in works on Differential Equations that the linear partial differential equation of the first order:

$$(1)\qquad P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}=R(x,y,z),$$

can be solved if the system of ordinary equations (§§ 196–197)

$$(2)\qquad \frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)},$$

can be solved. In fact, if the solutions of (2) can be written in the form $f(x, y, z) = \text{const.}$, $\phi(x, y, z) = \text{const.}$, then the general solution of (1) is $f(x, y, z) = A[\phi(x, y, z)]$, where A denotes, as in § 198, an arbitrary function. The truth of this fact can be established by a direct check in equation (1).

Example 1. Solve the equation $x^2(\partial z/\partial x) + xy(\partial z/\partial y) = z^2$.

The auxiliary equations (2) above become $dx/x^2 = dy/(xy) = dz/z^2$. These were solved in Ex. 1, § 197; the solutions may be written: $y/x = c_1$, $(z - x)/(xz) = c_2$. Hence the general solution of the given equation is $(z - x)/(xz) = A(y/x)$ or $z = x + xz\,A(y/x)$.

EXERCISES LXXXII.—PARTIAL DIFFERENTIAL EQUATIONS

1. Solve each of the following equations, where z represents a function of x and y:

(a) $\dfrac{\partial z}{\partial x}=c.$ (d) $\dfrac{\partial z}{\partial x}=x^2-y^2.$ (g) $\dfrac{\partial^2 z}{\partial x\,\partial y}=\dfrac{\partial z}{\partial x}.$

(b) $\dfrac{\partial^2 z}{\partial x^2}=0.$ (e) $\dfrac{\partial^2 z}{\partial x\,\partial y}=xy.$ (h) $\dfrac{\partial^3 z}{\partial x^2\,\partial y}=0.$

(c) $\dfrac{\partial^2 z}{\partial x\,\partial y}=0.$ (f) $\dfrac{\partial^2 z}{\partial y^2}=2x.$ (i) $\dfrac{\partial^3 z}{\partial x\partial y^2}=\dfrac{\partial^2 z}{\partial y^2}.$

2. Solve each of the following equations by comparison with a corresponding example (see § 199) of List LXXXI.

(a) $\dfrac{y^2}{x}\dfrac{\partial z}{\partial x}+\dfrac{x^2}{y}\dfrac{\partial z}{\partial y}=z.$ (c) $zy\dfrac{\partial z}{\partial x}+xz\dfrac{\partial z}{\partial y}=x+y.$

(b) $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}+z=0.$ (d) $(y+z)\dfrac{\partial z}{\partial x}+(x+z)\dfrac{\partial z}{\partial y}=x+y.$

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Greek Alphabet

LETTERS	NAMES	LETTERS	NAMES	LETTERS	NAMES	LETTERS	NAMES
Α α	Alpha	Η η	Eta	Ν ν	Nu	Τ τ	Tau
Β β	Beta	Θ θ	Theta	Ξ ξ	Xi	Υ υ	Upsilon
Γ γ	Gamma	Ι ι	Iota	Ο ο	Omicron	Φ φ	Phi
Δ δ	Delta	Κ κ	Kappa	Π π	Pi	Χ χ	Chi
Ε ε	Epsilon	Λ λ	Lambda	Ρ ρ	Rho	Ψ ψ	Psi
Ζ ζ	Zeta	Μ μ	Mu	Σ σ ς	Sigma	Ω ω	Omega

TABLES

[Roman page numbers refer to the body of the text; italic page numbers refer to these *Tables*.]

TABLE I

SIGNS AND ABBREVIATIONS

1. *Elementary signs assumed known without explanation:*

$+$; $\pm$; $\mp$; $-$; $=$; $a \times b = a \cdot b = ab$; $a \div b = a/b = a : b = \dfrac{a}{b}$; a^2; a^3; a^n; $a^{-n} = 1/a^n$; $a^{1/n} = \sqrt[n]{a}$; $a^{p/q} = \sqrt[q]{a^p}$; $a^0 = 1$; ( ); [ ]; { }; $\overline{\quad}$; a', a'', $\cdots$, $a^{(n)}$ (accents); a_1, a_2, $\cdots$, a_n (subscripts).

2. *Other elementary signs:*

$\neq$, not equal to.	$\geqq$, greater than or equal to.
$>$, greater than.	$\leqq$, less than or equal to.
$<$, less than.	$n!$ (or $\lfloor n$), factorial $n = n(n-1) \cdots 3 \cdot 2 \cdot 1$.
q.p., approximately.	$\lvert a \rvert$, absolute or numerical value of a.

3. *Signs peculiar to The Calculus and its Applications:*

(*a*) *Given a plane curve* $y = f(x)$ *in rectangular coördinates* (x, y);

m = slope $= dy/dx = f'(x) = y'$ = first derivative; see p. 23.

[Also occasionally $D_x y$, f_x, $\dot{y}$, p, by some writers.]

α = angle between positive x-axis and curve $= \tan^{-1} m$.

Δy, $\Delta^2 y$, $\cdots$, $\Delta^n y$, first, second, $\cdots$, n^{th} differences (or increments) of y.

$dy = f'(x) \cdot \Delta x$, $d^2 y = f''(x) \cdot \overline{\Delta x}^2$, $\cdots$, $d^n y = f^{(n)}(x) \cdot \overline{\Delta x}^n$, first, second, $\cdots$, n^{th} differentials of y.

r_r = relative rate of increase, or logarithmic derivative; see p. 146;

$= f'(x) \div f(x) = (dy/dx) \div y = d(\log y)/dx = r_p \div 100$.

r_p = percentage rate of increase $= 100 \cdot r_r$.

b = flexion $= d^2y/dx^2 = f''(x) = y''$ = second derivative; see p. 71.

$d^n y/dx^n = f^{(n)}(x) = y^{(n)} = n^{\text{th}}$ derivative.

K = curvature $= 1 \div R$; R = radius of curvature $= 1 \div K$; p. 170.

$\int f(x)\,dx$ = indefinite integral of $f(x)$; see p. 96.

$\int_a^b f(x)\,dx = \int_{x=a}^{x=b} f(x)\,dx$ = definite integral of $f(x)$; see p. 115.

s = length of arc ; $s\Big]_{x=a}^{x=b}$ = arc between $x = a$ and $x = b$.

$A\Big]_a^b = A\Big]_{x=a}^{x=b}$ = area between $y = 0$, $y = f(x)$, $x = a$, $x = b$; see p. 116.

(*b*) *Given a curve $\rho = f(\theta)$ in polar coördinates (ρ, θ) :*

$\psi = \angle$ (radius vector and curve) $= \text{ctn}^{-1}[(d\rho/d\theta) \div \rho]$
$= \text{ctn}^{-1}[d(\log \rho)/d\theta]$.

$\phi = \angle$ (circle about O and curve) $= \tan^{-1}[(d\rho/d\theta) \div \rho]$
$= \tan^{-1}[d(\log \rho)/d\theta]$.

$A\Big]_\alpha^\beta = A\Big]_{\theta=\alpha}^{\theta=\beta}$ = area between $\rho = f(\theta)$, $\theta = \alpha$, $\theta = \beta$; see p. 212.

(*c*) *For problems in plane motion :*

s = distance.
t = time.
m = mass.
v = speed.
$\boldsymbol{v}$ = velocity (vector).
$\boldsymbol{j}$ = acc. (vector).
θ = angle (of rotation).
ω = angular speed.

v_x = horizontal speed = projection of $\boldsymbol{v}$ on Ox.
v_y = vertical speed = projection of $\boldsymbol{v}$ on Oy.
j_x = horizontal acceleration = proj. of $\boldsymbol{j}$ on Ox.
j_y = vertical acceleration = proj. of $\boldsymbol{j}$ on Oy.
j_N = normal acc. = proj. of $\boldsymbol{j}$ on the normal.
j_T = tangential acc. = proj. of $\boldsymbol{j}$ on the tangent.
α = angular acceleration.
g = acceleration due to gravity.

(*d*) *Problems in space ; functions $z = f(x, y, \cdots)$ of several variables :*

Previous notations are generalized when possible without ambiguity, exceptions are
$$p = \partial x/\partial x = f_x\,; \quad q = \partial z/\partial y = f_y\,;$$
$$r = \partial^2 z/\partial x^2 = f_{xx}\,; \quad s = \partial^2 z/\partial x\,\partial y = f_{xy} = f_{yx}\,; \quad t = \partial^2 z/\partial y^2 = f_{yy}.$$

[The notation (dz/dx), used by some writers for $\partial z/\partial x$ is ambiguous.]

4 *Other letters commonly used with special meanings :*

π = ratio of circumference to diameter of circle = 3.14159⋯.
e = base of Napierian (or hyperbolic) logarithms = 2.71828⋯.
$M = \log_{10} e$ = modulus of Napierian to common logarithms = 0.434⋯

$\sum$ = "sum of such term as" ; thus : $\sum_{i=1}^{i=n} a_i^2 = a_1^2 + a_2^2 + \cdots a_n^2$.

(α, β, γ), — direction angles of a line in space.
(l, m, n), — direction cosines ; $l = \cos \alpha$, etc.
S. H. M. — simple harmonic motion.
ϵ or e, — eccentricity of a conic ; *also* phase angle of a S. H. M.

a, — amplitude of a S. H. M.
$(a.\ b)$, — semiaxes of a conic ; (a, b, c), semiaxes of a conicoid.
Δ = difference (of two values of a quantity).
ρ = density ; *also* radius vector, radius of curvature, radius of gyration.

5. *Trigonometric, logarithmic, hyperbolic, and other transcendental functions:* See *Tables*, II, A ; II, F, 3 ; II, G ; II, H ; and consult Index.

6. *Inverse function notations:*

If $y = f(x)$, then $f^{-1}(y) = x$; f^{-1} denotes an *inverse function*. [This notation is ambiguous ; confusion with $\{f(x)\}^{-1} = 1 \div f(x)$.]

$\sin^{-1} x$ or arc sin x, — inverse of sin x, or anti-sine of x, or **arc sine** $\boldsymbol{x}$, or angle whose sine is x. [Other inverse trigonometric functions, and hyperbolic functions, follow the same notations. See *Tables*, II, G, 18 ; H, 7.]

TABLE II

STANDARD FORMULAS

A. Exponents and Logarithms.

(The letters B, b, etc. indicate *base;* L, l, ⋯ indicate *logarithm;* N, n, ⋯ indicate *number;* base arbitrary when not stated. See § 73, p. 130.)

Laws of Exponents	Rules of Logarithms
(1) $N = B^L$; in particular $1 = B^0$; $B = B^1$; $1/B = B^{-1}$.	(1)′ $L = \log_B N$, *i.e.* $N = B^{\log_B N}$; and: $\log 1 = 0$; $\log_B B = 1$; $\log_B(1/B) = -1$.
(2) $B^L \cdot B^l = B^{L+l}$.	(2)′ $\log(N \cdot n) = \log N + \log n$.
(3) $B^L \div B^l = B^{L-l}$.	(3)′ $\log(N \div n) = \log N - \log n$.
(4) $(B^L)^n = B^{nL}$.	(4)′ $\log(N^n) = n \log N$.
(5) $N = B^L$, $B = b^k$, $N = b^{kL}$.	(5)′ $\log_b N = \log_b B \cdot \log_B N$.

$B=e$, $b=10$	gives $k=0.4342945=M=\log_{10} e$;	$\log_{10} N = M \cdot \log_e N$.
$B=10$, $b=e$	gives $k=2.302585=1 \div M=\log_e 10$;	$\log_e N=(1 \div M) \log_{10} N$.
$b=N$	gives $L=1/k$, $1=\log_b B \cdot \log_B b$;	*e.g.*, $\log_e 10=1 \div \log_{10} e$.
$L=x$	gives $10^x=e^{x \div M}$;	$e^x=10^{Mx}$.
$N=x$	gives $\log_{10} x=M \cdot \log_e x$;	$\log_e x=(1 \div M) \log_{10} x$.

(6) $y = cx^n$ gives $v = nu + k$, $u = \log_{10} x$, $v = \log_{10} y$, $k = \log_{10} c$.

(7) $y = ce^{ax}$ gives $v = mx + k$, $v = \log_{10} y$, $m = a \log_{10} e = aM$, $k = \log_{10} c$.

B. Factors.

(1) $a^2 - b^2 = (a-b)(a+b)$. (2) $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

(3) $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1})$.

(4) $a^{2n+1} + b^{2n+1} = (a+b)(a^{2n} - a^{2n-1}b + \cdots + b^{2n})$.

See also *Tables*, IV, Nos. 16, 20, 21, 49, 50.

(5) *Polynomials:* if $f(a) = 0$, $f(x)$ has a factor $x - a$; in general: $f(x) \div (x-a)$ gives remainder $f(a)$.

(6) $(a \pm b)^n = a^n \pm \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \cdots + (\pm 1)^n b^n$.
See II, **E**, 1, p. 7.

C. Solution of Equations.

(1) $ax^2 + bx + c = 0$, roots: $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{D}}{2a}$,
where

$D = b^2 - 4ac$; roots of (1) are $\left\{\begin{matrix}\text{real}\\ \text{coincident}\\ \text{imaginary}\end{matrix}\right\}$ when $D \left\{\begin{matrix}>0\\ =0\\ <0\end{matrix}\right\}$.

(2) $x^n + p_1 x^{n-1} + p_2 x^{n-1} + \cdots + p_{n-1}x + p_n = 0$. Roots: $x_1, x_2, \cdots, x_n$; then $\sum x_i = -p_1$, $\sum_{i \neq j} x_i x_j = p_2$, $\sum_{i \neq j \neq k} x_i x_j x_k = -p_3$, etc.

(3) $f(x) - \phi(x) = 0$: roots given by intersections of $y = f(x)$, $y = \phi(x)$. (Logarithmic chart often useful.) Find roots approximately; redraw figure on larger scale near intersection. (*Generalized Horner Process.*)

(4) *Simultaneous Equations:* $f(x, y) = 0$, $\phi(x, y) = 0$: roots (x, y) are points of intersection; redraw on larger scale as in (3).

(5) *Linear Equations:*

(*a*) 2 equations in 2 unknowns: $\left\{\begin{matrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{matrix}\right\}$.

Solutions: $x = \begin{vmatrix} c_1 b_1 \\ c_2 b_2 \end{vmatrix} \div \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} = (c_1b_2 - c_2b_1) \div (a_1b_2 - a_2b_1)$,

$$y = \begin{vmatrix} a_1 c_1 \\ a_2 c_2 \end{vmatrix} \div \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} = (a_1c_2 - a_2c_1) \div (a_1b_2 - a_2b_1).$$

(*b*) n equations in n unknowns: $a_i x_1 + b_i x_2 + \cdots + k_i x_n = p_i$; $i = 1, 2, \cdots, n$.

$$\text{Solutions: } x_i = \begin{vmatrix} a_1 \, b_1 \cdots p_1 \cdots k_1 \\ a_2 \, b_2 \cdots p_2 \cdots k_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n \, b_n \cdots p_n \cdots k_n \end{vmatrix} \div D \quad \left\{ \begin{matrix} \text{Column of } p\text{'s replaces column of} \\ \text{coefficients of } x_i. \end{matrix} \right\}$$

where

$$D = \begin{vmatrix} a_1 \, b_1 \cdots k_1 \\ a_2 \, b_2 \cdots k_2 \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ a_n \, b_n \cdots k_n \end{vmatrix} = a_1 \begin{vmatrix} b_2 \cdots k_2 \\ b_3 \cdots k_3 \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ b_n \cdots k_n \end{vmatrix} - a_2 \begin{vmatrix} b_1 \cdots k_1 \\ b_3 \cdots k_3 \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ b_n \cdots k_n \end{vmatrix} + \cdots + (-1)^{n-1} a_n \begin{vmatrix} b_1 \cdots k_1 \\ b_2 \cdots k_2 \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ b_{n-1} \; k_{n-1} \end{vmatrix}.$$

[Coefficient of a_i skips ith row of D. The last formula is a general definition of a **determinant.**]

D. Applications of Algebra.

1. *Interest.* (P = principal; p = rate per cent; $r = p \div 100$; n = number of years; A_n = amount after n years.)

(*a*) Simple interest: $A_n = P(1 + nr)$.

(*b*) Yearly compound interest: $A_n = P \cdot (1 + r)^n$.

(*c*) Semiannually compounded: $A_n = P(1 + r/2)^{2n}$.

(*d*) Compounded once each mth part of year: $A_n = P(1 + r/m)^{mn}$.

(*e*) Continuously compounded: $A_n = P \lim\limits_{m \doteq \infty} (1 + r/m)^{mn} = Pe^{nr}$.

2. *Annuities. Depreciation.* (I = yearly income (or depreciation or payment or charge); n = number of years annuity, or depreciation, runs.)

(*a*) Present worth P of yearly annuity I:

$$P = I[(1 + r)^n - 1] \div [r(1 + r)^n].$$

(*b*) Annuity I purchasable by present amount P; or, yearly depreciation I of plant of value P:

$$I = P[r(1 + r)^n] \div [(1 + r)^n - 1].$$

(*c*) *Final* value A_n of n yearly payments:

$$A_n = I(1 + r)[(1 + r)^n - 1] \div r.$$

3. *Permutations* $P_{n,r}$, *and Combinations* $C_{n,r}$, of n things r at a time, without repetitions:

(*a*) $P_{n,r} = n(n-1)\cdots(n-r+1) = n! \div (n-r)!$

(*b*) $C_{n,r} = P_{n,r} \div r! = [n(n-1)\cdots(n-r+1)] \div r!$

4. *Chance and Probability.*

(*a*) Chance of an event = (number of favorable cases) ÷ (total number of trials) $\leqq 1$.

Chance of successive (independent) events = product of separate chances $\leqq 1$.

Chance of at least one of several (independent) events = sum of separate chances.

(*b*) Probable value v of an observed quantity:

$v = \left(\sum m_i\right) \div n =$ arithmetic mean of n measurements m_1, m_2, $\cdots$, m_n; probable error in $v = \pm .6745\sqrt{\left(\sum(v-m_i)^2\right) \div n(n-1)}$.

(If the observations are unequally reliable, count each one a number of times, p_i, which represents its estimated reliability; p_i = "weight" of m_i).

(*c*) Probable value of k in formula $v = kx$:

$k = \sum x_i v_i \div \sum x_i^2$, from n measurements (x_1, v_1), (x_2, v_2), $\cdots$, (x_n, v_n); probable error in $k = \pm .6745\sqrt{\sum(kx_i - v_i)^2 \div (n-1)\sum x_i^2}$. See Exs. 4, p. 262; 29, p. 342.

(*d*) Probable values of k, l, m, $\cdots$, in formula $v = kx + ly + mz + \cdots$ are solutions of the equations:

$$\begin{array}{l}
k\sum x_i^2 + l\sum x_i y_i + m\sum x_i z_i + \cdots = \sum x_i v_i \\
k\sum x_i y_i + l\sum y_i^2 + m\sum y_i z_i + \cdots = \sum y_i v_i \\
k\sum x_i z_i + l\sum y_i z_i + m\sum z_i^2 + \cdots = \sum z_i v_i \\
\cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
\cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
\cdot \qquad \cdot \qquad \cdot \qquad \cdot
\end{array}$$

See also Exs. 18–23, p. 69, Example 2, p. 323, and Exs. 10–22, p. 328. (*Rules for Least Squares.* See also *Observational Errors*, No. III, J.)

E. Series.

1. *Binomial Theorem: Expansion of* $(a+b)^n$.

(*a*) n a positive integer: $(a+b)^n = a^n + \sum_{r=1}^{r=n} C_{n,r}a^{n-r}b^r$;

[$C_{n,r}$: see No. II, D, 3, p. 6, and also II, B, 6, p. 4.]

(*b*) n fractional or negative, $|a| > |b|$:

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + C_{n,r}a^{n-r}b^r + \cdots \text{ (forever).}$$

(*c*) Special cases:

$$(1 \pm x)^n = 1 \pm \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \cdots; \ (|x|<1).$$

$$\frac{1}{1 \pm x} = (1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp \cdots; \ (|x|<1). \quad \text{(Geometric progression.)}$$

$$\sqrt{1 \pm x} = (1 \pm x)^{1/2} = 1 \pm \frac{1}{2}x - \frac{1}{2^2 \cdot 2!}x^2 \pm \frac{1 \cdot 3}{2^3 \cdot 3!}x^3 - \cdots; \ (|x|<1).$$

$$\frac{1}{\sqrt{1 \pm x}} = (1 \pm x)^{-1/2} = 1 \mp \frac{1}{2}x + \frac{1 \cdot 3}{2^2 \cdot 2!}x^2 \mp \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}x^3 + \cdots; \ (|x|<1).$$

2. Arithmetic series: $a + (a+d) + (a+2d) + \cdots + (a+(n-1)d)$; last term $= l = a + (n-1)d$; sum $= s = n(a+l)/2$

3. Geometric series: $a + ar + ar^2 + ar^3 + \cdots$.

(*a*) n terms: $l = ar^{n-1}$; $s = \dfrac{rl - a}{r-1} = a\dfrac{r^n - 1}{r-1}$.

(*b*) infinite series, $|r| < 1$: $s = a/(1-r)$.

4. $1 + 2 + 3 + 4 + \cdots + (n-1) + n = n(n+1)/2$.

5. $2 + 4 + 6 + 8 + \cdots + (2n-2) + 2n = n(n+1)$.

6. $1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2$.

7. $1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = n(n+1)(2n+1) \div 3!$

8. $1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 = [n(n+1)/2]^2$.

9. $1 + 1/1! + 1/2! + 1/3! + \cdots = \lim_{n=\infty}\left(1 + \dfrac{1}{n}\right)^n = e = 2.71828\ldots$.

10. $e^x = 1 + x/1! + x^2/2! + x^3/3! \cdots$; (all x); $a^x = e^{x \log a}$.

11. $\log_e(1 \pm x) = \pm x - x^2/2 \pm x^3/3 - x^4/4 \pm x^5/5 - \cdots$; $(-1 < x < +1)$.

12. $\log_e[(1+x)/(1-x)] = 2[x + x^3/3 + x^5/5 + \cdots]$; $(-1 < x < +1)$.

[Computation of $\log N$: set $N = (1+x)/(1-x)$; then $x = (N-1)/(N+1)$; use II, A, 5′.]

13. $\sin x = x/1! - x^3/3! + x^5/5! - x^7/7! + \cdots$; (all x).

14. $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots$; (all x).

15. $\tan x = x + x^3/3 + 2x^5/15 + 17x^7/315 + \cdots$; $(|x| < \pi/2)$.

General term: $2^{2n}(2^{2n}-1)B_{2n-1} \div (2n)!$; see B_n, *Tables*, V, N, p. 58.

16. $\operatorname{ctn} x = 1/x - x/3 - x^3/45 - \sum 2B_{2n-1}(2x)^{2n-1} \div (2n)!$;
$(0 < |x| < \pi)$.

17. $\sec x = 1 + x^2/2! + 5x^4/4! + \sum [B_{2n}x^{2n}/(2n)!]$; $(|x| < \pi/2)$.

18. $\csc x = 1/x + x/3! + \sum [2(2^{2n+1}-1)B_{2n+1}x^{2n+1}/(2n+2)!]$;
$(0 < |x| < \pi)$.

19. $\sin^{-1} x = \pi/2 - \cos^{-1} x = x + x^3/(2\cdot 3) + 1\cdot 3x^5/(2\cdot 4\cdot 5) + \cdots$; $(|x| < 1)$.

20. $\tan^{-1} x = \pi/2 - \operatorname{ctn}^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + \cdots$; $(|x| < 1)$.

21. $(e^x + e^{-x})/2 = \cosh x = 1 + x^2/2! + x^4/4! + x^6/6! + \cdots$; (all x).

22. $(e^x - e^{-x})/2 = \sinh x = x + x^3/3! + x^5/5! + x^7/7! + \cdots$; (all x).

23. $e^{-x^2} = 1 - x^2 + x^4/2! - x^6/3! + x^8/4! - \cdots$; (all x).

24. $f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2! + \cdots$
$+ f^{n-1}(a)(x-a)^{n-1}/(n-1)! + E_n$.

Taylor's Theorem; *Remainder* E_n: $|E_n| \le [\text{Max.}\ |f^{(n)}(x)|]\,|(x-a)^n| \div n!$; $E_n = f^n[a + p(x-a)](x-a)^n \div n!$; $E_n = (1-p)^{n-1}f^{(n)}[a + p(x-a)](x-a)^n/n!$; $|p| < 1$.

Set $a = 0$: $f(x) = f(0) + f'(0)x + f''(0)x^2/2! + \cdots + f^{n-1}(0)x^{n-1}/(n-1)! + E_n$;
[*Maclaurin*].

Set $x = r + h$, $a = r$: $f(r+h) = f(r) + hf'(r) + h^2f''(r)/2! + \cdots + E_n$.

25. $f(x+h, y+k) = f(x, y) + [hf_x(x, y) + kf_y(x, y)]$
$+ [h^2f_{xx} + 2hkf_{xy} + k^2f_{yy}] \div 2! + \cdots + E_n$;

$|E_n| \le M(|h| + |k|)^n \div n!$, M = maximum of absolute values of all n^{th} derivatives.

26. If $f(x) = a_0/2 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots$
$+ b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots$; $(-\pi < x < +\pi)$.

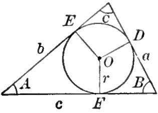
$$a_n = \frac{1}{\pi}\int_{x=-\pi}^{x=+\pi} f(x)\cos nx\,dx;\quad b_n = \frac{1}{\pi}\int_{x=-\pi}^{x=\pi} f(x)\sin nx\,dx.$$ **Fourier Theorem.**

F. Geometric Magnitudes. Mensuration.

l = length (or perimeter); A = area; V = volume.

DIMENSIONS OR EQUATIONS — FORMULAS

1. Triangle.



Sides: a, b, c. Angles: A, B, C.
Altitude from A on $a = h_a$.
$s = (a + b + c)/2$;
$\pi = A + B + C = 180°$;
$r = \sqrt{(s-a)(s-b)(s-c) \div s}$
$= (s - a) \tan (A/2)$;
$c = b \cos A + a \cos B$;
$c^2 = a^2 + b^2 - 2\,ab \cos C$.

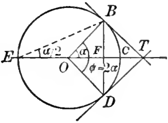
$l = a + b + c = 2\,s$;
$A = ah_a/2 = bh_b/2 = ch_c/2$
$= (1/2)\,ab \sin C$, etc.
$= rs = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$;
$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$;
$\tan \dfrac{B-C}{2} = \dfrac{b-c}{b+c} \operatorname{ctn} \dfrac{A}{2}$.

2. Trapezoid.

h = height. b_1, b_2 = bases.

$A = h\,(b_1 + b_2)/2$.

3. Circle.



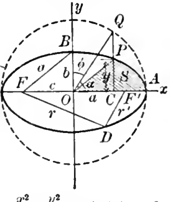
$\angle OEB = \alpha/2$;
$\angle OBT = 90°$;
$\angle FBT = \alpha$;
$\angle FBO = 90° - \alpha$
$= \angle FTB$;

r = radius; d = diameter;
$\alpha = COB$ at center
$= \dfrac{\text{arc } CB}{r}$ (radians)
$= \dfrac{180}{\pi}\,\dfrac{\text{arc } CB}{r}$ (degrees);
$\alpha/2 = CEB$, $\phi = 2\,\alpha = DOB$;
$\sin \alpha = FB \div r = 1 \div \csc \alpha$;
$\cos \alpha = OF \div r = 1 \div \sec \alpha$;
$\tan \alpha = TB \div r = 1 \div \operatorname{ctn} \alpha$;
$\operatorname{vers} \alpha = FC \div r = 1 - \cos \alpha$;
$\operatorname{ex\,sec} \alpha = CT \div r = \sec \alpha - 1$.

$l = 2\,\pi r = \pi d = 2\,A/r$;
$A = \pi r^2 = \pi d^2/4 = l\,r/2$;
arc $CB = r \cdot \alpha$, (α in radians)
$= \pi r \alpha/180$, (α in degrees);
Chord $DB = 2\,r \sin \alpha$
$= 2\,r \sin (\phi/2)$;
Sector $ODCB = \dfrac{\alpha}{180}\,\pi r^2$, ($\alpha$ in degrees);
Triangle $DOB = r^2 \sin \alpha \cos \alpha$
$= (1/2) r^2 \sin 2\,\alpha$;
Segment $DFBC = r^2\,[\pi\alpha/180 - (\sin 2\,\alpha)/2]$.

$\tan (\alpha/2) = BF \div EF = \sin \alpha/(1 + \cos \alpha)$;
$\sin (\alpha/2) = BF \div EB = \sqrt{(1 - \cos \alpha)/2}$;
$\cos (\alpha/2) = EF \div EB = \sqrt{(1 + \cos \alpha)/2}$.

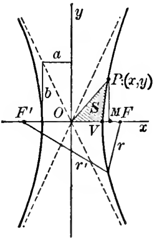
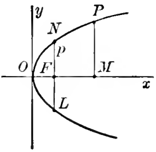
4. Ellipse. $e < 1$.



$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, (origin at O).
or $\rho = \dfrac{p}{1 - e \cos \theta}$, (pole at F);
Foci, F, F'; Center; O.

a, b, semiaxes; r, r', radii.
$c = \sqrt{a^2 - b^2}$;
$e = c/a = \sqrt{a^2 - b^2}/a$, (eccentricity);
$p = b^2/c = a(1 - e^2)/e$;
$\alpha = \tan^{-1}\left(\dfrac{ay}{bx}\right)$ = eccentric angle;
$x = a \cos \alpha$, $y = b \sin \alpha$;
$x = a \cos \dfrac{2\,S}{ab}$, $y = b \sin \dfrac{2\,S}{ab}$;
$x = a \sin \phi$, $y = b \cos \phi$;
$\phi = \pi/2 - \alpha$.

$r + r'$ = const. $= 2\,a$. $A = \pi ab$;
$S = OAP = \dfrac{ab}{2}\,\alpha = \dfrac{ab}{2} \cos^{-1} \dfrac{x}{a}$;
Arc $AP = \displaystyle\int_x^a \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}\,dx$ [*Tables.*]
$= a \displaystyle\int_0^\alpha \sqrt{1 - e^2 \cos^2 \alpha}\,d\alpha$;
where $\cos \alpha = x/a$.
Arc $BP = \displaystyle\int_0^x \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}\,dx$
$= aE\left(\dfrac{x}{a}, e\right)$
$= a \displaystyle\int_0^\phi \sqrt{1 - e^2 \sin^2 \phi}\,d\phi$
$\phi = \pi/2 - \alpha$; $\sin \phi = x/a$.

	Dimensions or Equations	Formulas
5. Hyperbola. $e>1$; 	a, b: semiaxes; r, r': radii; $c=\sqrt{a^2+b^2}$; $e=c/a=\sqrt{a^2+b^2}/a$; $p=b^2/c=a\,(e^2-1)/e$. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, or $\rho=\frac{p}{1-e\cos\theta}$, (origin at O) (pole at F).	$r'-r=\text{const.}=2\,a$; $S=\text{Sector } OVP=\frac{ab}{2}\log\left(\frac{x}{a}+\frac{y}{b}\right)$ $=\frac{ab}{2}\cosh^{-1}\left(\frac{x}{a}\right)=\frac{ab}{2}\sinh^{-1}\left(\frac{y}{b}\right)$; $x=a\cosh\frac{2S}{ab}$, $y=b\sinh\frac{2S}{ab}$; or if $\tan\phi=\sinh\frac{2S}{ab}$, $x=a\sec\phi$, $y=b\tan\phi$.
6. Parabola. $e=1$. 	$p=LN/2$; LN = latus rectum. $OF=p/2=LN/4$. $y^2=2\,px$, (origin at O); $\rho=\frac{p}{1-\cos\theta}$, (pole at F).	Area $ONPM=\frac{2}{3}\sqrt{2}\,x^{3/2}p^{1/2}$; Arc $OP=\int_{y=0}^{y=y}\sqrt{1+(y/p)^2}dy$. (See *Tables*, p. *38*, No. 45 (a).
7. Prism.	B = area of base; h = height.	$V=B\cdot h$.
8. Prismoid (§ 71, p. 125).	B = lower base (area); M = middle section; T = upper base; h = height.	$V=\frac{h}{6}(B+4\,M+T)$. (See also *Tables*, IV, G, p. *46*.)

[The volume of each of the solids mentioned below, except (16), follows this formula, though not all are prismoids.]

9. Pyramid (any sort).	A = area of base; h = height.	$V=A\cdot h/3$.
10. Right Circular Cylinder.	r = radius of base; h = height; B = base (area).	A (curved) $=2\,\pi rh$; A (total) $=2\,\pi rh+2\,\pi r^2$; $V=\pi r^2h=Bh$.

	Dimensions or Equations	Formulas
11. Right Circular Cone. See Fig. 21, p. 87. $\tan \alpha = r/h$; $\cos \alpha = h/s$; $\sin \alpha = r/s$.	r = radius of base; h = height; B = base; s = slant height; α = half vertex angle.	$s = \sqrt{r^2 + h^2}$; A (curved) $= \pi r \sqrt{r^2 + h^2} = \pi r s$; A (total) $= \pi r (s + r)$; $V = \pi r^2 h/3 = Bh/3$.
12. Frustum of Cone. B = lower base (area); T = upper base.	r = radius lower base; R = radius upper base; h = height; s = slant height.	$s = \sqrt{(R - r)^2 + h^2}$; A (curved) $= \pi s (R + r)$; $V = \pi h (R^2 + Rr + r^2)/3$.
13. Sphere. (*a*) *Entire Sphere.*	$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$; r = radius; d = diameter; C = great circle (area).	$A = 4\pi r^2 = \pi d^2 = 4C$; $V = 4\pi r^3/3 = \pi d^3/6 = A \cdot r/3 = 4Cr/3$.
(*b*) *Spherical Segment.* Other notations as above.	a = radius of base of segment; h = height of segment.	$a^2 = h(2r - h)$; $A = 2\pi r h = \pi (a^2 + h^2)$; $V = \pi h (3a^2 + h^2)/6 = \pi h^2 (3r - h)/3$.
(*c*) *Spherical Zone.*	h = height of zone; a, b = radii of bases.	$A = 2\pi r h$; $V = \pi h (3a^2 + 3b^2 + h^2)/6$.
(*d*) *Spherical Lune.*	α = angle of lune (degrees).	$A = \pi r^2 \alpha/90$.
(*e*) *Spherical Triangle* Sides α, β, γ. Angles A, B, C. $k = \sqrt{[\sin(s - \alpha)\sin(s - \beta)\sin(s - \gamma)]/\sin s}$; $K = \sqrt{-\cos S/[\cos(S - A)\cos(S - B)\cos(S - C)]}$.	$E = A + B + C - 180°$; $S = (A + B + C)/2$; $s = (\alpha + \beta + \gamma)/2$.	$A = \pi r^2 E/180$, $\frac{\sin A}{\sin \alpha} = \frac{\sin B}{\sin \beta} = \frac{\sin C}{\sin \gamma}$; $\cos \alpha = \cos \beta \cos \gamma + \sin \beta \sin \gamma \cos A$; $\cos A = -\cos B \cos C + \sin B \sin C \cos \alpha$; $\tan(A/2) = k/\sin(s - \alpha)$; $\tan(\alpha/2) = K \cos(S - A)$.
14. Ellipsoid. Semiaxes, a, b, c.	$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.	$V = 4\pi abc/3$.
15. Paraboloid of Revolution. $x^2 + y^2 = 2pz$.	r = radius of base; h = height.	$V = \pi r^2 h/2 = \pi p h^2$.
16. Anchor Ring. $\sqrt{x^2 + y^2} \pm \sqrt{r^2 - z^2} = R$.	r = radius, generating circle; R = mean radius of ring.	$A = 4\pi^2 Rr$; $V = 2\pi^2 Rr^2$.

[See also *Standard Applications of Integration*, *Tables*, IV, II, p. 46.]

G. Trigonometric Relations. For Trigonometric Mensuration Formulas, see II, F, 1, 3, 13 *e*, p. *9*.

1. *Definitions.* See also II, F, 3, p. *9*.

$$\sin A = y/r\ ;\ \cos A = x/r\ ;\ \tan A = y/x\ ;$$
$$\csc A = r/y\ ;\ \sec A = r/x\ ;\ \text{ctn}\ A = r/y\ ;$$
$$\text{vers}\ A = 1 - \cos A\ ;\ \text{exsec}\ A = \sec A - 1.$$

2. *Special Values, Signs, etc, for sine, cosine, and tangent.*

Angle	0°	30°	45°	60°	90°	180°	270°	360° ± A or 0° ± A	90° ± A	180° ± A	270° ± A
sin	± 0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	± 0	-1	$\pm \sin A$	$+\cos A$	$\mp \sin A$	$-\cos A$
cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	± 0	-1	± 0	$+\cos A$	$\mp \sin A$	$-\cos A$	$\pm \sin A$
tan	± 0	$\sqrt{3}/3$	1	$\sqrt{3}$	$\pm\infty$	± 0	$\pm\infty$	$\pm \tan A$	$\mp \text{ctn}\ A$	$\pm \tan A$	$\mp \text{ctn}\ A$

[± 0 and $\pm\infty$ indicate that the function changes sign.]

3. $\csc A = 1/\sin A$; $\sec A = 1/\cos A$; $\tan A = 1/\text{ctn}\ A$.

4. $x^2 + y^2 = r^2$: $\cos^2 A + \sin^2 A = 1$; $1 + \tan^2 A = \sec^2 A$; $\text{ctn}^2 A + 1 = \csc^2 A$.

5. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.

6. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$.

7. $\tan(A \pm B) = [\tan A \pm \tan B] \div [1 \mp \tan A \tan B]$.

8. $\sin 2A = 2 \sin A \cos A$; $\sin \alpha = 2 \sin(\alpha/2) \cos(\alpha/2)$.

9. $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$;
$\cos \alpha = \cos^2(\alpha/2) - \sin^2(\alpha/2)$; see also II, F, 3, p. *9*.

10. $\sin 3A = 3 \sin A - 4 \sin^3 A$. 11. $\cos 3A = 4 \cos^3 A - 3 \cos A$.

12. $\tan 2A = 2 \tan A \div [1 - \tan^2 A]$. [See also II, F, 3, p. *9*].

13. $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$;
$\sin \alpha \pm \sin \beta = 2 \sin[(\alpha \pm \beta)/2] \cos[(\alpha \mp \beta)/2]$.

14. $2 \cos A \cos B = \cos(A - B) + \cos(A + B)$;
$\cos \alpha + \cos \beta = 2 \cos[(\alpha + \beta)/2] \cos[(\alpha - \beta)/2]$.

15. $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$;
$\cos \alpha - \cos \beta = -2 \sin[(\alpha + \beta)/2] \sin[(\alpha - \beta)/2]$.

16. $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B)\sin(A-B)$.

17. $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B)\cos(A-B)$.

18. *Definitions of Inverse Trigonometric Functions:*

(*a*) $y = \sin^{-1} x$ = arc sin x = angle whose sine is x, if $x = \sin y$; usually y is selected in 1st or 4th quadrant].

(*b*) $y = \cos^{-1} x$ = arc cos x, if $x = \cos y$; [take y in 1st or 2d quadrant].

(*c*) $y = \tan^{-1} x$ = arc tan x, if $x = \tan y$; [take y in 1st or 4th quadrant].

19. $\sin^{-1} x = \pi/2 - \cos^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}[x/\sqrt{1-x^2}]$

$$= \csc^{-1}(1/x) = \sec^{-1}[1/\sqrt{1-x^2}] = \text{ctn}^{-1}[\sqrt{1-x^2}/x].$$

20. $\cos^{-1} x = \pi/2 - \sin^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}[\sqrt{1-x^2}/x]$

$$= \sec^{-1}(1/x) = \csc^{-1}[1/\sqrt{1-x^2}] = \text{ctn}^{-1}[x/\sqrt{1-x^2}].$$

21. $\tan^{-1} x = \pi/2 - \text{ctn}^{-1} x = \text{ctn}^{-1}(1/x) = \sin^{-1}[x/\sqrt{1+x^2}]$

$$= \cos^{-1}[1/\sqrt{1+x^2}] = \sec^{-1}\sqrt{1+x^2} = \csc^{-1}[\sqrt{1+x^2}/x].$$

22. *Special values, correct quadrants, etc., for inverse functions.*

VALUE	+	−	0	1	−1	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	>1	−*k*
$\sin^{-1}x$	1st *Q*	4th *Q*	0	$\pi/2$	$-\pi/2$	$\pi/6$	$\pi/4$	$\pi/3$	0.62	——	$-\sin^{-1}(+k)$
$\cos^{-1}x$	1st *Q*	2d *Q*	$\pi/2$	0	π	$\pi/3$	$\pi/4$	$\pi/6$	0.96	——	$\pi-\cos^{-1}(+k)$
$\tan^{-1}x$	1st *Q*	4th *Q*	0	$\pi/4$	$-\pi/4$	0.46	0.62	0.71	$\pi/6$	$>\pi/4$	$-\tan^{-1}(+k)$

H. Hyperbolic Functions.

1. *Definitions.* (See figures III, E, J_2, pp. *20*, *28*; and V, C, p. ***52***.

$\sinh x = (e^x - e^{-x})/2$; $\cosh x = (e^x + e^{-x})/2$;

$\tanh x = \sinh x/\cosh x = (e^x - e^{-x})/(e^x + e^{-x})$;

$\text{ctnh}\, x = 1/\tanh x$; $\text{sech}\, x = 1/\cosh x$; $\text{csch}\, x = 1/\sinh x$.

ϕ = Gudermannian of $x = gd\, x = \tan^{-1}(\sinh x)$; $\tan\phi = \sinh x$.

$$= \tan^{-1}[(e^x - e^{-x})/2] = 2\tan^{-1}e^x - \pi/2$$

2. $\cosh^2 x - \sinh^2 x = 1$. **3.** $1 - \tanh^2 x = \text{sech}^2 x$.

4. $1 - \text{ctnh}^2 x = \text{csch}^2 x$.

5. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$.

6. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$.

7. $y = \sinh^{-1} x = \arg \sinh x =$ inverse hyperbolic sine, if $x = \sinh y$. [Similar inverse forms corresponding to $\cosh x$, $\tanh x$, etc.]

8. $\sinh^{-1} x = \cosh^{-1} \sqrt{x^2+1} = \operatorname{csch}^{-1}(1/x) = \log(x + \sqrt{x^2+1})$.

9. $\cosh^{-1} x = \sinh^{-1} \sqrt{x^2-1} = \operatorname{sech}^{-1}(1/x) = \log(x + \sqrt{x^2-1})$.

10. $\tanh^{-1} x = \operatorname{ctnh}^{-1}(1/x) = (1/2)\log[(1+x)/(1-x)]$.

11. If $\phi = gd\, x$, $\sinh x = \tan\phi$, $\cosh x = \operatorname{ctn}\phi$, $\tanh x = \sin\phi$.

I. Analytic Geometry

[(x, y) or (a, b) denote a point; (x_1, y_1) and (x_2, y_2) two points; etc.]

1. Distance $l = P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2}$.

2. Projection of P_1P_2 on $Ox = \Delta x = x_2 - x_1 = l \cos\alpha$, where
$\alpha = \angle(Ox,\ P_1P_2)$.

3. Projection of P_1P_2 on $Oy = \Delta y = y_2 - y_1 = l \sin\alpha$.

4. Slope of $P_1P_2 = \tan\alpha = (y_2 - y_1)/(x_2 - x_1) = \Delta y/\Delta x$.

5. Division point of P_1P_2 in ratio r: $(x_1 + r\,\Delta x,\ y_1 + r\,\Delta y)$.

6. Equation $Ax + By + C = 0$: straight line.

(*a*) $y = mx + b$: slope, m; y-intercept, b.

(*b*) $y - y_0 = m(x - x_0)$: slope, m; passes through (x_0, y_0).

(*c*) $(y-y_1)/(y_2-y_1) = (x-x_1)/(x_2-x_1)$: passes through (x_1, y_1), (x_2, y_2).

(*d*) $x \cos\alpha + y\cos\beta = p$: distance to origin, p; $\alpha = \angle(Ox,\ n)$; $\beta = \angle(Oy,\ n)$; n = normal through origin.

[General equation $Ax + By + C = 0$ reduces to this on division by $\sqrt{A^2+B^2}$.]

7. Angle between lines of slopes m_1, $m_2 = \tan^{-1}[(m_1 - m_2)/(1 + m_1m_2)]$. [Parallel, if $m_1 = m_2$; perpendicular, if $1 + m_1m_2 = 0$, *i.e.* if $m_1 = -1/m_2$.]

8. Transformation $x = x' + h$, $y = y' + k$. [Translation to (h, k).]

9. Transformation $x = cx'$, $y = ky'$. [Increase of scale in ratio c on x-axis; in ratio k on y-axis.]

10. Transformation, $x = x'\cos\theta - y'\sin\theta$, $y = x'\sin\theta + y'\cos\theta$. [Rotation of axes through angle θ.]

11. Transformation to polar coördinates (ρ, θ): $x = \rho\cos\theta$, $y = \rho\sin\theta$. Reverse transformation: $\rho = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$.

12. *Circle:* $(x-a)^2+(y-b)^2=r^2$; center, (a, b); radius, r; or $(x-a)=r\cos\theta$, $(y-b)=r\sin\theta$. (θ variable.)

13. *Parabola:* $y^2=2px$: vertex at origin; latus rectum $2p$.

14. *Ellipse:* $x^2/a^2+y^2/b^2=1$: center at origin; semiaxes, a, b. (See II, F, 4, p. *9*.)

15. *Hyperbola:* $x^2/a^2-y^2/b^2=1$; center at origin; semiaxes, a, b; asymptotes, $x/a \pm y/b=0$. See II, F, 5, p. *10*.

(*a*) If $a=b$, $x^2-y^2=a^2$; retangular hyperbola.

(*b*) $xy=k$, rectangular hyperbola; asymptotes: the axes.

(*c*) $y=(ax+b)/(cx+d)$, rectangular hyperbola; asymptotes: $x=-d/c$, $y=a/c$.

16. *Parabolic Curves:* $y=a_0+a_1x+a_2x^2+\cdots+a_nx^n$.

[Graph of polynomial; see also Figs. A, B, pp. *17*, *18*.]

17. Lagrange Interpolation Formula. **Given** $\boldsymbol{y=f(x)}$**,** the polynomial approximation of degree $n-1$ [parabolic curve through n points, (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n)] is

$$\boldsymbol{y=P(x)=y_1p_1(x)+y_2p_2(x)+\cdots+y_np_n(x),}$$

where the polynomials $p_1(x)$, $p_2(x)$, $\cdots$, $p_n(x)$ are

$$p_i(x)=\frac{(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

[Numerator skips $(x-x_i)$; denominator skips (x_i-x_i). Proof by direct check.]

[For a variety of other curves, see *Tables*, III, pp. *17–32*.]

Formulas of Solid Geometry, §§ 162–3; pp. 315–19; see also Figs. III, N_1–N_5, p. 31. When possible the preceding formulas of plane geometry are so phrased that an additional term of the kind indicated gives the analogous formula of solid geometry. In particular, see 6 *d*, p. *14*.

J. Differential Formulas.

[See (*a*) *List of Differential Formulas of Elementary Functions*, pp. 40, 173.

(*b*) *List of Standard Integrals*, p. 174, *and Tables*, IV, p. *33*. Reverse these to obtain Differential formulas.

(*c*) *List of Standard Applications of Integration*, *Tables*, IV, II, p. *46*.

(*d*) *Infinite Series*, *Taylor's Formula*, etc., see *Tables*, II, E, pp. *7–8*.

1. $\boldsymbol{y = f(x)}$: $dy = f'(x)\,dx$, $f'(x) = dy \div dx = dy/dx$.

2. $\boldsymbol{F(x, y) = 0}$: $F_x\,dx + F_y\,dy = 0$, or $dy = -[F_x \div F_y]\,dx$; $F_x = \partial F/\partial x$, $F_y = \partial F/\partial y$.

3. $\boldsymbol{x = f(t),\ y = \phi(t)}$:

(*a*) $dx = f'(t)\,dt$, $dy = \phi'(t)\,dt$, $dy/dx = \phi'(t) \div f'(t)$.

(*b*) $d^2y/dx^2 = d[dy/dx]/dx = d[\phi' \div f']/dx = [\phi''f' - f''\phi'] \div (f')^3$.

(*c*) $d^3y/dx^3 = d[d^2y/dx^2]/dx = d[(\phi''f' - f''\phi') \div (f')^3]/dt \div f'$.

4. *Transformation* $\boldsymbol{x = f(t)}$: $y = \phi(x)$ becomes $y = \phi(f(t)) = \psi(t)$.

(*a*) dy/dx becomes $dy/dt \div f'(t)$; [see 3 (*a*)].

(*b*) d^2y/dx^2 becomes $[(d^2y/dt^2)\cdot f'(t) - (dy/dt)f''(t)] \div [f'(t)]^3$; [see 3 (*b*)].

5. *Transformation* $\boldsymbol{x = f(t, u),\ y = \phi(t, u)}$: $y = F(x)$ becomes $u = \Phi(t)$.

(*a*) dy/dx becomes $\dfrac{dy}{dt} \div \dfrac{dx}{dt}$ or $\left[\dfrac{\partial\phi}{\partial t} + \dfrac{\partial\phi}{\partial u}\cdot\dfrac{du}{dt}\right] \div \left[\dfrac{\partial f}{\partial t} + \dfrac{\partial f}{\partial u}\cdot\dfrac{du}{dt}\right]$.

(*b*) d^2y/dx^2 becomes $d[dy/dx]/dt \div dx/dt$; [compute as in 5 (*a*)].

6. *Polar Transformation* $x = \rho\cos\theta$, $y = \rho\sin\theta$.

$$dx = \cos\theta\,d\rho - \rho\sin\theta\,d\theta;\quad dy = \sin\theta\,d\rho + \rho\cos\theta\,d\theta,$$
$$d^2x = \cos\theta\,d^2\rho - 2\sin\theta\,d\rho\,d\theta - \rho\cos\theta\,d\theta^2,$$
$$d^2y = \sin\theta\,d^2\rho + 2\cos\theta\,d\rho\,d\theta - \rho\sin\theta\,d\theta^2.$$

7. $\boldsymbol{z = F(x, y)}$: $dz = F_x\,dx + F_y\,dy = p\,dx + q\,dy$; [see I, 3 (*d*), p. 2].

8. *Transformation* $x = f(u, v)$, $y = \phi(u, v)$: $z = F(x, y) = \Phi(u, v)$.

(*a*) $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial f}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial\phi}{\partial u}$, $\quad \dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial f}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial\phi}{\partial v}$.

(*b*) $\dfrac{\partial z}{\partial x} = A\dfrac{\partial z}{\partial u} + B\dfrac{\partial z}{\partial v}$, $\quad \dfrac{\partial z}{\partial y} = C\dfrac{\partial z}{\partial u} + D\dfrac{\partial z}{\partial v}$.

[A, B, C, D found by solving 8 (*a*) for $\partial z/\partial x$ and $\partial z/\partial y$.]

(*c*)
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right)$$
$$= A\frac{\partial}{\partial u}\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right) + B\frac{\partial}{\partial v}\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right).$$

[Similar expressions for $\partial^2 z/\partial y^2$ and higher derivatives.]

TABLE III

STANDARD CURVES

A. Curves $y = x^n$, all pass through $(1, 1)$; positive powers also through $(0, 0)$; negative powers asymptotic to the y-axis. Special cases: $n = 0, 1$

Chart of $y = x^n$ for positive, negative, fractional values of n

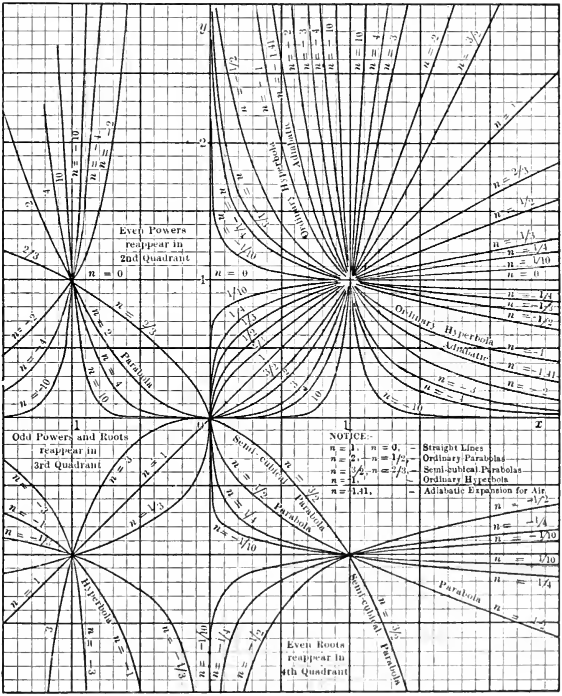


FIG. A

are **straight lines**; $n = 2$, $1/2$ are ordinary **parabolas**; $n = -1$ is an ordinary **hyperbola**; $n = 3/2$, $2/3$ are **semi-cubical parabolas.**

The curves $pv^m = c$ occur in the theory of gas expansion, where p = pressure; v = volume; c and m constants. In *isothermal* expansion (p. 292) $m = 1$, whence $pv = c$ or $p = cv^{-1}$; ($n = -1$ in Fig. A). Choose scales so that $y = p/c$ and $v = x$. In ***adiabatic expansion of air***, $m = 1.41$ (nearly). The *area* $\int_{v_1}^{v_2} p\,dv$ = *work done* in compressing the gas from any given volume v_2 to a volume v_1.

B. Logarithmic Paper; Curves $y = x^n$, $y = kx^n$. Logarithmic paper is used chiefly in *experimental determination of the constants* k and n; and for *graphical tables.* In Fig. B, $k = 1$ except where given.

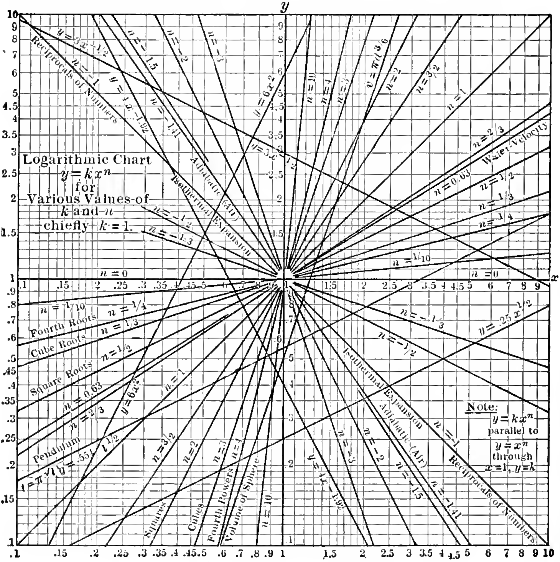


FIG. B

[See § 122, p. 229, above; also Williams-Hazen, *Hydraulic Tables;* Trautwine, *Engineers' Handbook;* D'Ocagne, *Nomographie.*] The line $y = x^{-1}$ gives the *reciprocals of numbers* by direct readings.

C. Trigonometric Functions. The **inverse trigonometric functions** are given by reading y first.

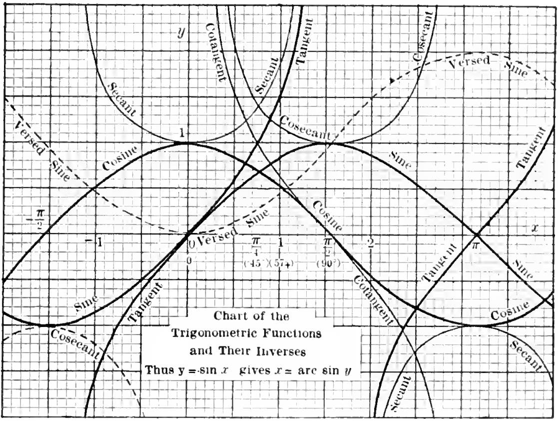


FIG. C

D. Logarithms and Exponentials: $y = \log_{10} x$ and $y = \log_e x$.
Note $\log_e x = \log_{10} x \ \log_e 10 = 2.303 \log_{10} x$. The values of the **exponential** functions $x = 10^y$ and $x = e^y$ are given by reading y first. See E.

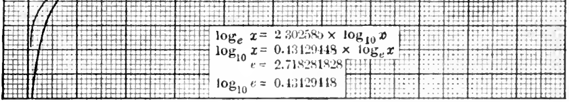


FIG. D

E. Exponential and Hyperbolic Functions. The **catenary** (hyperbolic cosine) [$y = \cosh x = (e^x + e^{-x})/2$] and the **hyperbolic sine** [$y = \sinh x = (e^x - e^{-x})/2$] are shown in their relation to the **exponential curves** $y = e^x$, $y = e^{-x}$. Notice that both hyperbolic curves are asymptotic to $y = e^x/2$.

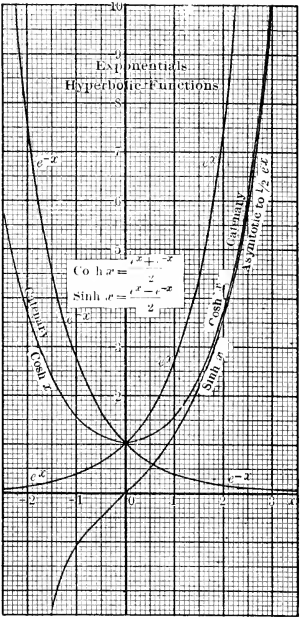


Fig. E

The curve $y = e^{-x}$ is the standard **damping** curve ; see Fig. F_2, and § 92, p. 160.

The **general catenary** is $y = (a/2)(e^{x/a} + e^{-x/a}) = a \cosh (x/a)$; it is the curve in which a flexible inelastic cord will hang. (Change the scale from 1 to a on both axes.)

F. Harmonic Curves. The general type of **simple harmonic curve** is $y = a \sin(kx + \epsilon)$:

CURVE	$a \sin(kx + \epsilon)$	$\sin x$	$\cos x$	$\sin 2x$	$(1/2) \sin(6x - 1.2)$
amplitude	a	1	1	1	1/2
wave-length	$2\pi/k$	2π	2π	π	$\pi/3$
phase	$-\epsilon/k$	0	$\pi/2$	0	0.2

A **compound harmonic curve** is formed by superposing simple harmonics: in Fig. F_1, $y = \sin 2x + (1/2) \sin(6x - 1.2)$ is drawn.

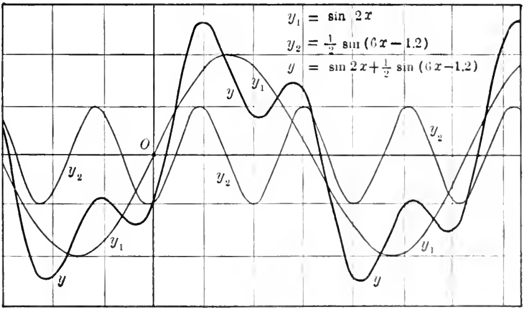


FIG. F_1

Such curves occur in theories of vibrations, sound, electricity. See §§ 90–92, pp. 157 160.

The simplest type of **damped vibrations** is $y = e^{-cx} \sin kx$: Fig. F_2 shows $y = e^{-x/2} \sin 3x$. The general form is $y = ae^{-cx} \sin(kx + \epsilon)$. Such damped simple vibrations may be superposed on other damped or undamped vibrations. See §§ 92, 189, pp. 160, 368.

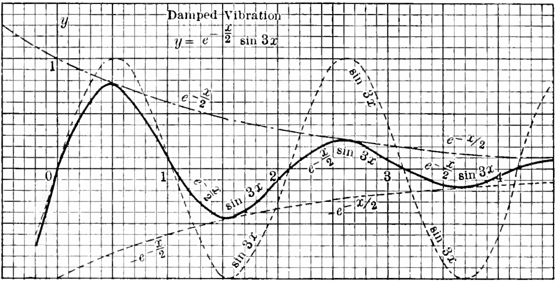


FIG. F_2

G. The Roulettes.

A roulette is the path of any point rigidly connected with a moving curve which rolls without slipping on another (fixed) curve.

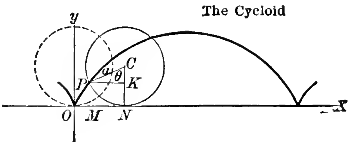


FIG. G_1

Figure G_1 shows the ordinary **cycloid**, a roulette formed by a point P on the rim of a wheel of radius a, which rolls on a straight line OX. See also Fig. 69, p. 307. The equations are

$$\begin{cases} x = OX - MN = a\theta - a \sin\theta, \\ y = NC - KC = a - a\cos\theta, \end{cases}$$

where $\theta = \angle NCP$.

Figure G_2 shows the curves traced by a point on a spoke of the wheel of Fig. H, or the spoke produced. These are called **trochoids**; their equations are

$$\begin{cases} x = a\theta - b\sin\theta, \\ y = a - b\cos\theta, \end{cases}$$

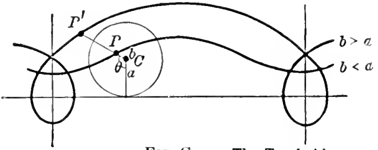
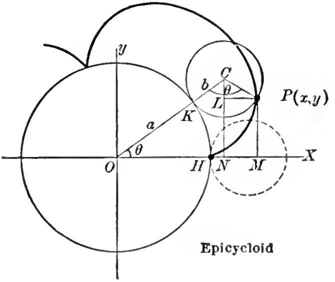


FIG. G_2 **The Trochoids**

where b is the distance PC. If $b > a$, the curve is called an **epitrochoid**; if $b < a$, a **hypotrochoid**.

Figure G_3 shows the **epicycloid**;

$$\begin{cases} x = (a+b)\cos\theta - b\cos\left[\dfrac{a+b}{b}\theta\right], \\ y = (a+b)\sin\theta - b\sin\left[\dfrac{a+b}{b}\theta\right], \end{cases}$$



Epicycloid

FIG. G_3

formed by a point on the circumference of a circle of radius b rolling on the exterior of a circle of radius a.

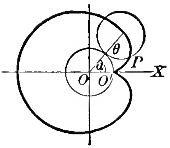


FIG. G_4

Figure G_4 shows the *special epicycloid*, $a = b$,

$$\begin{cases} x = 2\,a\cos\theta - a\cos 2\,\theta, \\ y = 2\,a\sin\theta - a\sin 2\,\theta, \end{cases}$$

which is called the **cardioid**; its equation in polar coördinates $(\rho,\ \phi)$ with pole at O' is $\rho = 2\,a\,(1 - \cos\phi)$.

Figure G_5 shows the hypocycloid:

$$\begin{cases} x = (a - b)\cos\theta + b\cos\left[\dfrac{a-b}{b}\,\theta\right], \\ y = (a - b)\sin\theta - b\sin\left[\dfrac{a-b}{b}\,\theta\right], \end{cases}$$

formed by a point on the circumference of a circle of radius b rolling on the interior of a circle of radius a.

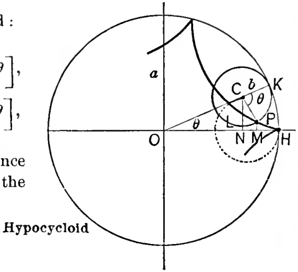


FIG. G_5

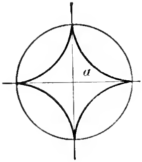


FIG. G_6

Figure G_6 shows the *special hypocycloid*, $a = 4\,b$,

$$\begin{cases} x = a\cos^3\theta, \\ y = a\sin^3\theta, \end{cases} \quad \text{or} \quad x^{2/3} + y^{2/3} = a^{2/3},$$

which is called the **four-cusped hypocycloid**, or **astroid**.

H. The Tractrix. This curve is the path of a particle P drawn by a cord PQ of fixed length a attached to a point Q which moves along the x-axis from 0 to $\pm\infty$. Its equation is

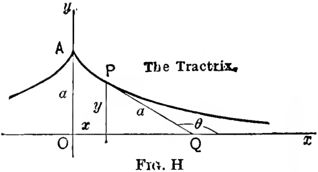


FIG. H

$$x = a\log\frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}.$$

I. Cubic and Quartic Curves.

Figure I_1 shows the **contour lines** of the surface $z = x^3 - 3x - y^2$ cut out by the planes $z = k$, for $k = -6, -4, -2, 0, 2, 4$; that is, the **cubic curves** $x^3 - 3x - y^2 = k$.

The surface has a maximum at $x = -1$, $y = 0$; the point $x = 1$, $y = 0$ is also a critical point, but the surface cuts through its tangent plane there, along the curve $k = -2$; $y^2 = x^3 - 3x + 2$.

These curves are drawn by means of the auxiliary curve $q = x^2 - 3x$, itself a type of cubic curve; then $y = \sqrt{q - k}$ is readily computed.

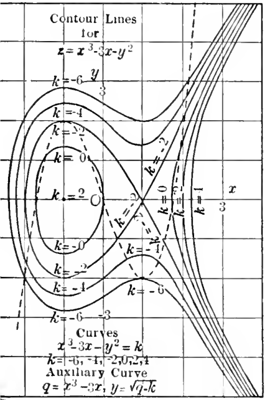


FIG. I_1

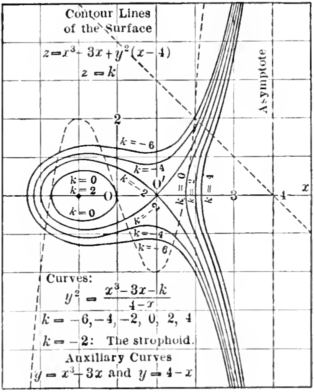


FIG. I_2

Figure I_2 shows the **contour lines** of the surface $z = x^3 - 3x + y^2(x - 4)$ for $z = k = -6, -4, -2, 0, 2, 4$; that is, the **cubic curves**

$$y^2 = (x^3 - 3x - k)/(4 - x).$$

The surface has a maximum at $(-1, 0)$. At $(1, 0)$ the horizontal tangent plane $z = -2$ cuts the surface in the **strophoid** $y^2 = (x^3 - 3x + 2)/(4 - x)$ whose equation with the new origin O' is $y^2 = x'^2(3 + x')/(3 - x')$. The line $x = 4$ is an asymptote for each of the curves.

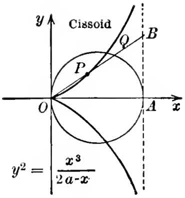


FIG. I_3

Figure I_3 shows another cubic: the **cissoid**, famous for its use in the ancient problem of the "duplication of the cube." Its equation is

$$y^2 = \frac{x^3}{2a - x}, \quad \text{or} \quad \rho = 2a \tan\theta \sec\theta.$$

It can be drawn by using an auxiliary curve as above; or by means of its geometric definition: $OP = QB$, when Oy and AB are vertical tangents to the circle OQA.

Figure I_4 shows the **conchoid** of Nicomedes, used by the ancients in the problem of trisection of an angle. Its equation is

$$y^2 = -x^2 + \left(\frac{bx}{x-a}\right)^2, \quad \text{or} \quad \rho = a\sec\theta \pm b.$$

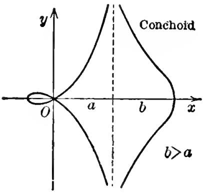


FIG. I_4

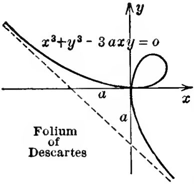


FIG. I_5

Figure I_5 shows the cubic $x^3 + y^3 - 3axy = 0$, called the **Folium of Descartes**; see Exs. 1, p. 46; 11–12, p. 63.

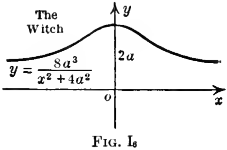


FIG. I_6

Figure I_6 shows the **witch of Agnesi**: $y = 8a^3/(x^2 + 4a^2)$; see Exs. 3, p. 163; 5 (*b*), 5 (*d*), p. 166; 5, p. 180; and see III, J, below.

Figure I_7 shows the **Cassinian ovals**, defined geometrically by the equation $PF \cdot PF_1 = k^2$; or by the **quartic** equation,

$$[(x-a)^2+y^2]\,[(x+a)^2+y^2]=k^4,$$

where $a = OF$ $(=1$ in Fig. $I_7)$. The special oval $k^2 = a^2$ is called the **lemmiscate,** $(x^2+y^2)^2 = 2\,a^2(x^2-y^2)$ or $\rho^2 = 2\,a^2 \cos 2\,\theta$.

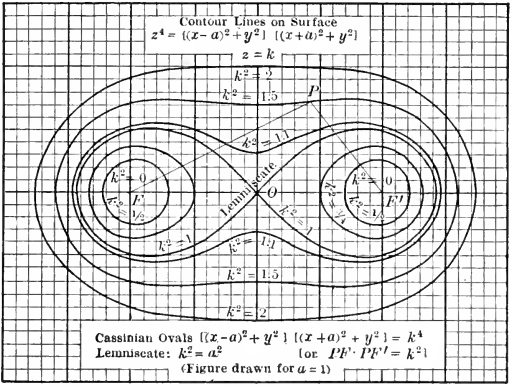


FIG. I_7

The ovals are also the *contour lines* of the surface

$$z^4=[(x-a)^2+y^2]\,[(x+a)^2+y^2],$$

which has minima at $(x = \pm\, a,\ y = 0)$, and a critical point with no extreme at origin.

J. Error or Probability Curves.

Figure J_1 is the so-called **curve of error**, or probability curve

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2},$$

where h is the *measure of precision*. See *Tables*, IV, H, 148, p. *48*; and V, G, p. *54*.

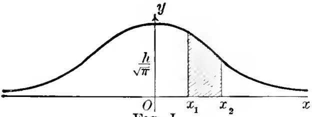


FIG. J_1

Figure J_2 shows the very similar curve

$$y = \text{sech}\, x = 2/(e^x + e^{-x}).$$

In some instances this curve, or the witch (Fig. I_6), may be used in place of Fig. J_1. Any of these curves, on a proper scale, give good approximations to the probable **distribution of any accidental data** which tend to group themselves about a mean.

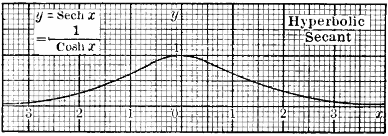


FIG. J_2

K. Polynomial Approximations.

Figure K_1 shows the first **Taylor polynomial approximations** to the function $y = \sin x$. (See § 134, p. 258.)

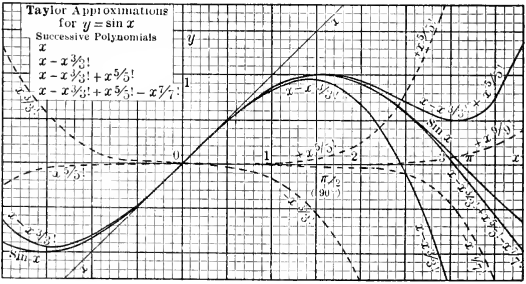


FIG. K_1

Figure K_2 shows the **Simpson-Lagrange approximations**: (1) by a broken line; (2) by an ordinary parabola; (3) by a cubic, which however degenerates into a parabola in this example. (Lagrange Interpolation Formula, *Tables*, p. *15*.)

The fourth approximation is so close that it cannot be drawn in the figure. In practice, the division points are taken closer together than is feasible in a figure.

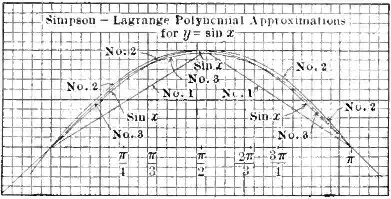


FIG. K_2

No. 1. $y = \frac{2}{\pi}x,\ 0 \le x \le \frac{\pi}{2};\ y = -\frac{2}{\pi}(x - \pi),\ \frac{\pi}{2} \le x \le \pi.$

No. 2. $y = \frac{4}{\pi}x - \frac{4}{\pi^2}x^2 = 1.273\,x - .405\,x^2,\ 0 \le x \le \pi.$

(Division points: $x_0 = 0,\ x_1 = \pi/2,\ x_2 = \pi$.)

No. 3. $y = \frac{9\sqrt{3}}{4\pi}x - \frac{9\sqrt{3}}{4\pi^2}x^2 - 0 \cdot x^3 = 1.245\,x - 0.395\,x^2.$

(Division points: $x_0 = 0,\ x_1 = \pi/3,\ x_2 = 2\pi/3,\ x_3 = \pi$.)

No. 4. $y = 15.38\,x - 7.642\,x^2 - 2.36\,x^3 + 0.376\,x^4.$

(Agrees too closely to show in the figure.)

L. Trigonometric Approximations.

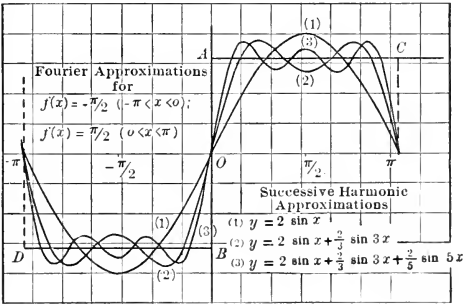
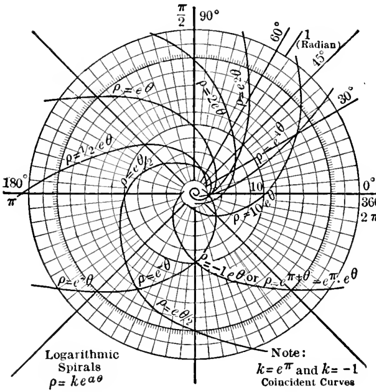


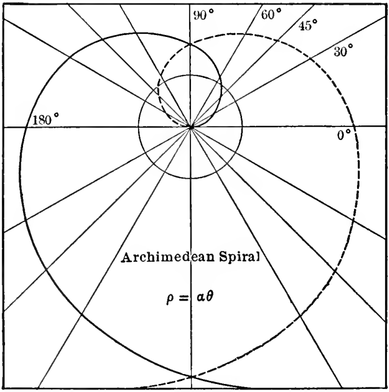
FIG. L

Figure L shows the approximation to the two detached line-segments $y = -\pi/2,\ (-\pi < x < 0),\ y = \pi/2,\ (0 < x < \pi)$ by means of an expression of the form $a_0 + a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$. See II, E, 26.



$\frac{\pi}{2}$ 90°
60°
1 (Radian)
45°
30°
$\rho = e\theta$
$\rho = 2e\theta$
$\rho = e^{2\theta}$
$\rho = e^{4\theta}$
$\rho = 1/2\, e\theta$
$\rho = e\theta/2$
180°
π
10
0°
360°
2π
$\rho = 10e\theta$
$\rho = -1e\theta$ or $\rho = e^{\pi+\theta} = e^{\pi} \cdot e^{\theta}$
$\rho = e^{-\theta}$
$\rho = e^{-\theta}$
$\rho = e\theta/2$
Logarithmic Spirals
$\rho = ke^{a\theta}$
Note:
$k = e^{\pi}$ and $k = -1$
Coincident Curves

FIG. M_1



90°
60°
45°
30°
180°
0°
Archimedean Spiral
$\rho = a\theta$

FIG. M_2

M. Spirals.

Figure M_1 shows the **logarithmic** [or **equiangular spirals** $\rho = ke^{a\theta}$] for several values of k and a. Note that $k = e^{\pi}$ and $k = -1$ give *the same curve.* (See § 96, p. 168.)

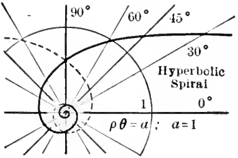
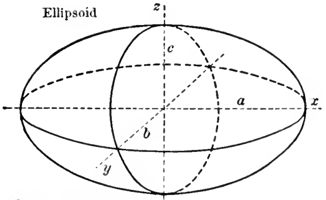


FIG. M_3

Figures M_2 and M_3 represent the Archimedean Spiral $\rho = a\theta$, and the **Hyperbolic Spiral** $\rho\theta = a$, respectively.

N. Quadric Surfaces.

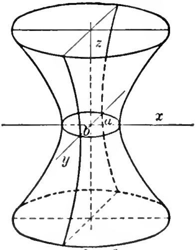
These are standard figures of the usual equations.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

FIG. N_1

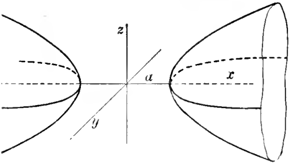
Hyperboloid of one Sheet



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

FIG. N_2

Hyperboloid of two Sheets



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

FIG. N_3

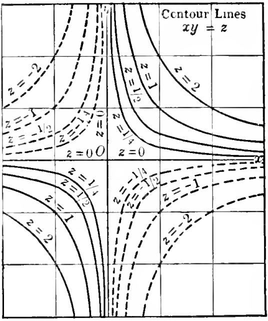
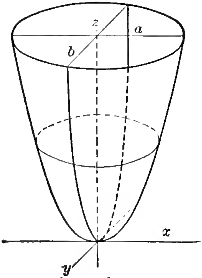


FIG. N_6

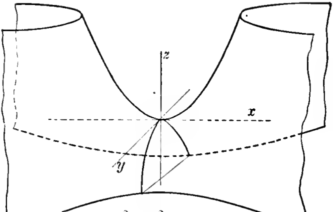
Elliptic Paraboloid



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = kz$$

FIG. N_4

Hyperbolic Paraboloid



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2\,kz$$

FIG. N_5

TABLE IV

STANDARD INTEGRALS

Index:

A. Fundamental General Formulas, p. *33*.

B. Integrand — Rational Algebraic, p. *34*.

C. Integrand Irrational, p. *37*.

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(*b*) Quadratic radical $\sqrt{\pm x^2 \pm a^2}$, p. *37*.

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A. Fundamental General Formulas.

1. **If** $\frac{du}{dx} = \frac{dv}{dx}$, **then** $u = v +$ **constant.** [Fundamental Theorem.]

2. **If** $\int u\,dx = I$, **then** $\frac{dI}{dx} = u$. [General Check.]

3. $\int cu\,dx = c\int u\,dx$.

4. $\int [u + v]\,dx = \int u\,dx + \int v\,dx$.

5. $\int u\,dv = uv - \int v\,du$. [Parts.]

6. $\left[\int f(u)\,du\right]_{u=\phi(x)} = \int f[\phi(x)]\,\frac{d\phi(x)}{dx}\,dx$. [Substitution.]

B. Integrand — Rational Algebraic.

7. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}$, $n \neq -1$, see 8.

NOTES. (*a*) $\int$(Any Polynomial) dx, — use 3, 4, 7.

(*b*) $\int$(Product of Two Polynomials) dx, — expand, then use 3, 4, 7.

(*c*) $\int c\,dx = cx$, by 3, 7.

8. $\int \dfrac{dx}{x} = \log_e x = (\log_{10} x)(\log_e 10) = (2.302585)\log_{10} x.$

NOTES. (*a*) $\int (1/x^m)\,dx$, — use 7 with $n = -m$ if $m \neq 1$; use 8 if $m = 1$.

(*b*) $\int$[(Any Polynomial)$/x^m$] dx, — use short division, then 7 and 8.

9. $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\arctan\dfrac{x}{a} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} = \dfrac{1}{a}\operatorname{ctn}^{-1}\dfrac{a}{x}$

$= -\dfrac{1}{a}\operatorname{ctn}^{-1}\dfrac{x}{a}$ [+ const.].

10. $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\log\dfrac{x-a}{x+a} = \dfrac{1}{2a}\log\dfrac{a-x}{a+x}$ [+ const.].

NOTE. *All rational functions are integrated by reductions to* 7, 8, 9. The reductions are performed by 3, 4, 6. No. 10 and all that follow are results of this process.

11. $\int (ax+b)^n\,dx = \dfrac{1}{a}\dfrac{(ax+b)^{n+1}}{n+1}$, $n \neq -1$. (See No. 12.) [From 7.]

12. $\int \dfrac{dx}{(ax+b)} = \dfrac{1}{a}\log(ax+b)$. [From 8.]

NOTES. (*a*) $\int \dfrac{Ax+B}{ax+b}\,dx$, — use long division, then 7 and 12.

(*b*) $\int \dfrac{\text{Any Polynomial}}{ax+b}\,dx$, — use long division, then 7 and 12.

13. $\int \dfrac{dx}{(ax+b)^m} = \dfrac{1}{a}\dfrac{-1}{(m-1)(ax+b)^{m-1}}$, $(m \neq 1)$. [From 11.]

14. $\int \dfrac{x\,dx}{(ax+b)^2} = \dfrac{1}{a^2}\left[\dfrac{b}{ax+b} + \log(ax+b)\right]$. [From 11, 12.]

NOTES. (*a*) $\int \dfrac{Ax+B}{(ax+b)^2}\,dx$, — combine A times No. 14 and B times No. 13, $m = 2$.

(*b*) $\int \dfrac{\text{(Any Polynomial)}}{(ax+b)^2}\,dx$, — use long division, then 7 and 14 (*a*); or use 15

15. $\left[\int F(x, ax+b)\,dx\right]_{u=ax+b} = \frac{1}{a}\int F\left(\frac{u-b}{a}, u\right) du.$ [From 6.]

NOTES. (*a*) *Restatement:* put u for $ax+b$, $\frac{u-b}{a}$ for x, $\frac{du}{a}$ for dx.

(*b*) $\int [x/(ax+b)^3]\,dx$, — use 15. *Ans.* $\frac{1}{a^2}\left[-\frac{1}{u}+\frac{b}{2u^2}\right]_{u=a+bx}$.

(*c*) $\int \frac{\text{(Any Polynomial)}}{(ax+b)^3}dx$, — use 15; then 8 (*b*).

(*d*) $\int \frac{\text{(Any Polynomial)}}{(ax+b)^m}dx$, — use 15 unless $m<3$, ; but see 12 (*b*), 14 (*b*).

(*e*) $\int x^n(ax+b)^m\,dx$, — use 15 if $m>n$; use 7 (*b*) if $m \leqq n$; see also 51–54.

16. $$\frac{1}{(ax+b)(cx+d)} = \frac{1}{ad-bc}\left[\frac{a}{ax+b}-\frac{c}{cx+d}\right].$$

NOTES. (*a*) $\int \frac{dx}{(ax+b)(cx+d)}$, — use 16, then 12. Special cases, — see 10 and 16 (*b*).

(*b*) $\int \frac{dx}{x(ax+b)} = \frac{1}{b-a}\int\left[\frac{1}{x}-\frac{a}{ax+b}\right]dx.$ (Special case of 16 (*a*).)

(*c*) $\int \frac{Ax+B}{(ax+b)(cx+d)}dx$, — use 16, then long division, 12.

(*d*) $\int \frac{\text{(Any Polynomial)}}{(ax+b)(cx+d)}dx$, — use 16, then long division, 7, 12.

(*e*) If $ad-bc=0$, 18 can be used.

17. $$\left[\int F(x, ax+b)\,dx\right]_{u=\frac{ax+b}{x}} = \int F\left(\frac{b}{u-a}, \frac{bu}{u-a}\right)\cdot\frac{-b\,du}{(u-a)^2}.$$

NOTES. (*a*) *Restatement:*

Put u for $\frac{ax+b}{x}$; $\frac{b}{u-a}$ for x; $\frac{bu}{u-a}$ for $ax+b$; $\frac{-b\,du}{(u-a)^2}$ for dx.

(*b*) $\int \frac{dx}{x^n(ax+b)^m} = -\frac{1}{b^{m+n-1}}\int\frac{(u-a)^{m+n-2}}{u^m}du$; then use 8 *b*.

(*c*) $\int \frac{dx}{x^2(ax+b)} = -\frac{u-a\log u}{b^2}$. (*d*) $\int\frac{dx}{x^3(ax+b)} = -\frac{u^2-4au+2a^2\log u}{2b^3}$.

18. $$\int \frac{dx}{ax^2+b} = \frac{1}{\sqrt{ab}}\tan^{-1}x\sqrt{\frac{a}{b}}, \text{ if } a>0,\ b>0. \text{ [See 9.]}$$

$$= \frac{1}{2\sqrt{-ab}}\log\frac{\sqrt{a}\,x-\sqrt{-b}}{\sqrt{a}\,x+\sqrt{-b}}, \text{ if } a>0,\ b<0. \text{ [See 10.]}$$

NOTES. (*a*) $\int\frac{dx}{ax^2-c}$, — use 18 (2nd part); $b=-c$. (*b*) $\int\frac{dx}{c-ax^2} = -\int\frac{dx}{ax^2-c}$.

19. $\displaystyle\int \frac{x\,dx}{ax^2+b} = \frac{1}{2a}\log(ax^2+b).$

NOTES. (*a*) $\displaystyle\int \frac{x^2\,dx}{ax^2+b}$, — use long division, then 18.

(*b*) $\displaystyle\int \frac{Ax+B}{ax^2+b}\,dx$, — use 18, 19.

(*c*) $\displaystyle\int \frac{\text{(Any Polynomial)}}{ax^2+b}\,dx$, — use long division, then 18, 19.

20. $$\frac{1}{(mx+n)(ax^2+b)} = \frac{1}{an^2+bm^2}\left[\frac{m^2}{mx+n} - a\,\frac{mx-n}{ax^2+b}\right].$$

NOTES. (*a*) $\displaystyle\int \frac{1}{(mx+n)(ax^2+b)}\,dx$, — use 20, then 12, 18, 19.

(*b*) $$\frac{Ax^2+Bx+C}{(mx+n)(ax^2+b)} = \frac{A}{a}\,\frac{1}{mx+n} + \frac{B}{m}\,\frac{1}{ax^2+b} + \left(C - \frac{Ab}{a} - \frac{Bn}{m}\right)\frac{1}{(mx+n)(ax^2+b)}.$$

(*c*) $\displaystyle\int \frac{\text{Any Polynomial}}{(mx+n)(ax^2+b)}\,dx$, — use long division, then 20 *b*, 12, 18, 20 *a*.

21. $$ax^2+bx+c = a\left[x+\frac{b}{2a}\right]^2 - \frac{b^2-4ac}{4a}.$$

NOTES. (*a*) $\displaystyle\int \left[\frac{dx}{ax^2+bx+c}\right]_{u=x+\frac{b}{2a}} = \int \frac{du}{au^2 - \dfrac{b^2-4ac}{4a}}$, then 18.

(*b*) $$\left[\int F(x,\ ax^2+bx+c)\,dx\right]_{u=x+\frac{b}{2a}} = \int F\left(u-\frac{b}{2a},\ au^2-\frac{b^2-4ac}{4a}\right)du.$$

(*c*) $\displaystyle\int \frac{\text{Any Polynomial}}{ax^2+bx+c}\,dx$, — long division, then 7, 21, 21 *b*, 18, and 19.

(*d*) $\displaystyle\int \frac{\text{Any Polynomial}}{\text{Any Cubic}}\,dx$, — long division, then find one real factor of cubic, then use 21, 21 *b*. [If the cubic has a double factor, set u = that factor, then use 17 *c*.]

22. $$\int \frac{x\,dx}{(ax^2+b)^2} = -\frac{1}{2a}\,\frac{1}{ax^2+b}.$$

23. $$\int \frac{dx}{(ax^2+b)^2} = \frac{x}{2b(ax^2+b)} + \frac{1}{2b}\int \frac{dx}{ax^2+b};\ \text{then 18.}$$

24. $$\int \frac{x\,dx}{(ax^2+b)^m} = \left[\frac{1}{2a}\int \frac{du}{u^m}\right]_{u=ax^2+b};\ \text{then 7 or 8.}$$

25. $$\int \frac{dx}{(ax^2+b)^m} = \frac{1}{2b(m-1)}\,\frac{x}{(ax^2+b)^{m-1}} + \frac{2m-3}{2(m-1)b}\int \frac{dx}{(ax^2+b)^{m-1}}$$

NOTES. (*a*) Use 25 repeatedly to reach 23 and thence 18.

(*b*) Final forms in **partial fraction** reduction are of types 12, 24, 25 (by use of 21).

C. (a) Integrand Irrational : involving $r = \sqrt{ax+b}$.

26. $\left[\int F(x, \sqrt{ax+b})\,dx\right]_{r=\sqrt{ax+b}} = \int F\left(\frac{r^2-b}{a}, r\right)\frac{2r}{a}\,dr.$

27. $\int \sqrt{ax+b}\,dx = \int r\frac{2r}{a}\,dr = \frac{2}{3a}r^3,\ r = \sqrt{ax+b}.$

28. $\int x\sqrt{ax+b}\,dx = \frac{2}{a^2}\int (r^4 - br^2)\,dr = \frac{2r^3}{a^2}\left[\frac{r^2}{5} - \frac{b}{3}\right].$

29. $\int \frac{dx}{\sqrt{ax+b}} = \frac{2}{a}\int dr = \frac{2}{a}r.$

30. $\int \frac{dx}{x\sqrt{ax+b}} = \int \frac{2\,dr}{r^2-b}$; use 9 or 10.

31. $\int \frac{dx}{x^2\sqrt{ax+b}} = 2a\int \frac{dr}{(r^2-b)^2}$; use 23.

NOTE. $\sqrt{ax+b}^3 = (ax+b)\sqrt{ax+b}$; $(\sqrt{ax+b})^3 = (ax+b)\sqrt{ax+b}$.

(b) Integrand Irrational: involving $\sqrt{\pm x^2 \pm a^2}$.

32. $\int \frac{dx}{\sqrt{a^2-x^2}} = \text{arc}\sin\frac{x}{a} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a} + [\text{const.}].$

33. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2}) = \sinh^{-1}\frac{x}{a}\ [+\text{const.}]$ for $+$,

or $\cosh^{-1}\frac{x}{a}\ [+\text{const.}]$ for $-$.

34. $\int \frac{dx}{\sqrt{2ax - x^2}} = \sin^{-1}\left(\frac{x-a}{a}\right) = -\cos^{-1}\frac{x-a}{a}\ [+\text{const.}]$

$= \text{vers}^{-1}x + \text{const.}$

35. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} = \frac{1}{a}\cos^{-1}\frac{a}{x} = -\frac{1}{a}\csc^{-1}\frac{x}{a}\ [+\text{const.}].$

36. $\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}.$

37. $\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}.$

38. $\int x\sqrt{a^2-x^2}\,dx = \frac{-1}{3}(\sqrt{a^2-x^2})^3.$

39. $\int x\sqrt{x^2+a^2}\,dx = \frac{1}{3}(\sqrt{x^2+a^2})^3.$

NOTES. (*a*) 32 and 33 furnish the basis for all which follow.
(*b*) 36, 37, 38, 39 follow from $x\,dx = d(x^2 + \text{const.})/2$.

40. $$\int \frac{x^2dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

41. $$\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left[\frac{a+\sqrt{a^2 \pm x^2}}{x}\right].$$

42. $$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{1}{a^2x}\sqrt{a^2-x^2}.$$

43. $$\sqrt{a^2-x^2} = \frac{a^2-x^2}{\sqrt{a^2-x^2}}.$$

NOTES. (to 43) (a) $\int \sqrt{a^2-x^2}\,dx = a^2\int \frac{dx}{\sqrt{a^2-x^2}} - \int \frac{x^2\,dx}{\sqrt{a^2-x^2}}$, then 32, 40.

(b) $\int \frac{\sqrt{a^2-x^2}}{x}dx = a^2\int \frac{dx}{x\sqrt{a^2-x^2}} - \int \frac{x\,dx}{\sqrt{a^2-x^2}}$, then 36, 41.

(c) $\int \frac{\sqrt{a^2-x^2}}{x^2}dx = a^2\int \frac{dx}{x^2\sqrt{a^2-x^2}} - \int \frac{dx}{\sqrt{a^2-x^2}}$, then 42, 32.

44. $$\int \frac{x^2dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log(x+\sqrt{x^2 \pm a^2}).$$

45. $$\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2x}.$$

46. $$\sqrt{x^2 \pm a^2} = \frac{x^2 \pm a^2}{\sqrt{x^2 \pm a^2}}.$$

NOTES. (to 46) (a) $\int \sqrt{x^2 \pm a^2}\,dx = \int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} \pm a^2\int \frac{dx}{\sqrt{x^2 \pm a^2}}$, then 44, 33.

(b) $\int \frac{\sqrt{x^2 \pm a^2}}{x}dx = \int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} \pm a^2\int \frac{dx}{x\sqrt{x^2 \pm a^2}}$, then 37, 35, or 41.

47. $$\int \frac{dx}{(\sqrt{a^2-x^2})^3} = \frac{x}{a^2\sqrt{a^2-x^2}}.$$

48. $$\int \frac{dx}{(\sqrt{x^2 \pm a^2})^3} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}.$$

NOTES. *Trigonometric Substitutions.* If the desired form is not found in 32–48, try 79. Then use Nos. 55–79, see 79. (b) See also D, 51–54, below.

49. $$\sqrt{\pm(ax^2+bx+c)} = \sqrt{a}\sqrt{\pm\left(x+\frac{b}{2a}\right)^2 \mp \frac{b^2-4ac}{4a^2}}.$$

NOTES. *Forms containing* $\sqrt{\pm(ax^2+bx+c)}$:

(a) $\int F(x, \sqrt{\pm(ax^2+bx+c)})\,dx$, — use 49, then put $u = x + b/2a$ [see 49 (b)]; then use 32–48.

(b) Remember $\sqrt{\pm(ax^2+bx+c)^3} = \pm(ax^2+bx+c)\sqrt{\pm(ax^2+bx+c)}$.

$$\sqrt{\pm(ax^2+bx+c)} = \pm(ax^2+bx+c) \div \sqrt{\pm(ax^2+bx+c)}.$$

(c) Simplify all radicals **first**.

50. $\sqrt{\dfrac{ax+b}{cx+d}} = \dfrac{ax+b}{\sqrt{(ax+b)(cx+d)}} = \dfrac{\sqrt{(ax+b)(cx+d)}}{cx+d}.$

NOTES. (*a*) Integrals containing $\sqrt{(ax+b)/(cx+d)}$: use 50, then 49, then 32–48.
(*b*) Substitution of $u = \sqrt{(ax+b)/(cx+d)}$ is successful without 50.

D. Integrals of Binomial Differentials — Reduction Formulas.

Symbols : $u = ax^n + b$; a, b, p, m, n, any numbers for which no denominator in the formula vanishes.

51. $\displaystyle\int x^m(ax^n+b)^p\,dx = \frac{1}{m+np+1}\left[x^{m+1}u^p + npb\int x^m u^{p-1}\,dx\right].$

52. $\displaystyle\int x^m(ax^n+b)^p\,dx$
$\displaystyle = \frac{1}{bn(p+1)}\left[-x^{m+1}u^{p+1} + (m+n+np+1)\int x^m u^{p+1}\,dx\right].$

53. $\displaystyle\int x^m(ax^n+b)^p\,dx$
$\displaystyle = \frac{1}{(m+1)b}\left[x^{m+1}u^{p+1} - a(m+n+np+1)\int x^{m+n}u^p\,dx\right].$

54. $\displaystyle\int x^m(ax^n+b)^p\,dx$
$\displaystyle = \frac{1}{a(m+np+1)}\left[x^{m-n+1}u^{p+1} - (m-n+1)b\int x^{m-n}u^p\,dx\right].$

NOTES. (*a*) These **reduction formulas** useful when p, m, or n are **fractional**; hence applications to **Irrational Integrands**.
(*b*) Repeated application may reduce to one of 32–48.
(*c*) Do not apply if p, m, n, are all integral, unless $n \geqq 2$ and p large. Note 11, 15, 17–25.

E$_a$. Integrand Transcendental : Trigonometric Functions.

55. $\displaystyle\int \sin x\,dx = -\cos x.$

56. $\displaystyle\int \sin^2 x\,dx = -\tfrac{1}{2}\cos x\sin x + \tfrac{1}{2}x = -\tfrac{1}{4}\sin 2x + \tfrac{1}{2}x.$

NOTE. $\int \sin^2 kx\,dx$, — set $kx = u$, and use 56. Likewise in 55–78.

57. $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$

NOTE. If n is odd, put $\sin^2 x = 1 - \cos^2 x$ and use 62.

58. $\int \cos x\, dx = \sin x.$

59. $\int \cos^2 x\, dx = \frac{1}{2}\sin x \cos x + \frac{1}{2}x = \frac{1}{4}\sin 2x + \frac{1}{2}x.$

60. $\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$

NOTE. If n is odd, put $\cos^2 x = 1 - \sin^2 x$ and use 63.

61. $\int \sin x \cos x\, dx = -\frac{1}{4}\cos 2x = \frac{1}{2}\sin^2 x\,[+\text{const.}].$

62. $\int \sin x \cos^n x\, dx = -\frac{\cos^{n+1} x}{n+1},\ n \neq -1.$

63. $\int \sin^n x \cos x\, dx = \frac{\sin^{n+1} x}{n+1},\ n \neq -1.$

64. $$\int \sin^n x \cos^m x\, dx = \frac{\sin^{n+1} x \cos^{m-1} x}{m+n} + \frac{m-1}{m+n}\int \sin^n x \cos^{m-2} x\, dx$$
$$= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2} x \cos^m x\, dx.$$

NOTE. If n is an odd integer, set $\sin^2 x = 1 - \cos^2 x$ and use 62. If m is odd, use 63.

65. $$\int \sin(mx)\cos(nx)\, dx = -\frac{\cos[(m+n)x]}{2(m+n)} - \frac{\cos[(m-n)x]}{2(m-n)},\quad m \neq \pm n.$$

66. $$\int \sin(mx)\sin(nx)\, dx = \frac{\sin[(m-n)x]}{2(m-n)} - \frac{\sin[(m+n)x]}{2(m+n)},\ m \neq \pm n.$$

67. $$\int \cos(mx)\cos(nx)\, dx = \frac{\sin[(m-n)x]}{2(m-n)} + \frac{\sin[(m+n)x]}{2(m+n)},\ m \neq \pm n.$$

68. $\int \tan x\, dx = -\log\cos x.$ 69. $\int \tan^2 x\, dx = \tan x - x.$

70. $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx.$

71. $\int \operatorname{ctn} x\, dx = \log \sin x.$ 72. $\int \operatorname{ctn}^2 x\, dx = -\operatorname{ctn} x - x.$

73. $\int \operatorname{ctn}^n x\, dx = -\frac{\operatorname{ctn}^{n-1} x}{n-1} - \int \operatorname{ctn}^{n-2} x\, dx.$

74. $\int \sec x\, dx = \log\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \log(\sec x + \tan x)\,[+\text{const.}].$

75. $\int \csc x\,dx = \log \tan \frac{x}{2} = -\log(\csc x + \operatorname{ctn} x)[+\text{const.}]$.

76. $\int \sec^2 x\,dx = \tan x$.

77. $\int \csc^2 x\,dx = -\operatorname{ctn} x$.

78. $\int \sec^m x \csc^n x\,dx = \int \frac{dx}{\sin^n x \cos^m x}$ (See also 64.)

$$= \frac{1}{m-1}\sec^{m-1} x \csc^{n-1} x + \frac{m+n-2}{m-1}\int \sec^{m-2} x \csc^n x\,dx$$

$$= -\frac{1}{n-1}\sec^{m-1} x \csc^{n-1} x + \frac{m+n-2}{n-1}\int \sec^m x \csc^{n-2} x\,dx.$$

NOTES. (*a*) In 64 and 78 and many others, m and n may have negative values.

(*b*) To reduce $\int [\sin^n x/\cos^m x]\,dx$ take m negative in 64.

(*c*) To reduce $\int [\cos^m x/\sin^n x]\,dx$ take n negative in 64.

79. **Substitutions** :

	$u =$	du	$\sin x$	$\cos x$	$\tan x$	x	dx
(1)	$\sin x$	$\cos x\,dx$	u	$\sqrt{1-u^2}$	$\frac{u}{\sqrt{1-u^2}}$	$\sin^{-1} u$	$\frac{du}{\sqrt{1-u^2}}$
(2)	$\cos x$	$-\sin x\,dx$	$\sqrt{1-u^2}$	u	$\frac{\sqrt{1-u^2}}{u}$	$\cos^{-1} u$	$-\frac{du}{\sqrt{1-u^2}}$
(3)	$\tan x$	$\sec^2 x\,dx$	$\frac{u}{\sqrt{1+u^2}}$	$\frac{1}{\sqrt{1+u^2}}$	u	$\tan^{-1} u$	$\frac{du}{1+u^2}$
(4)	$\sec x$	$\sec x \tan x\,dx$	$\frac{\sqrt{u^2-1}}{u}$	$\frac{1}{u}$	$\sqrt{u^2-1}$	$\sec^{-1} u$	$\frac{du}{u\sqrt{u^2-1}}$
(5)	$\tan \frac{x}{2}$	$\frac{1}{2}\sec^2 \frac{x}{2}\,dx$	$\frac{2u}{1+u^2}$	$\frac{1-u^2}{1+u^2}$	$\frac{2u}{1-u^2}$	$2\tan^{-1} u$	$\frac{2\,du}{1+u^2}$

Replace $\operatorname{ctn} x$, $\sec x$, $\csc x$ by $1/\tan x$, $1/\cos x$, $1/\sin x$, respectively

NOTES. (*a*) $\int F(\sin x)\cos x\,dx$, — use 79, (1).

(*b*) $\int F(\cos x)\sin x\,dx$, — use 79, (2).

(*c*) $\int F(\tan x)\sec^2 x\,dx$, — use 79, (3).

(*d*) **Inspection** of this table shows *desirable substitutions* from trigonometric to algebraic, and conversely. Thus, if only $\tan x$, $\sin^2 x$, $\cos^2 x$ appear, use 79, (3).

80. $$\int \frac{dx}{a + b \sin x} = \frac{1}{\sqrt{a^2 - b^2}} \sin^{-1} \frac{b + a \sin x}{a + b \sin x}, \text{ if } a^2 > b^2;$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \frac{b - \sqrt{b^2 - a^2} + a \tan (x/2)}{b + \sqrt{b^2 - a^2} + a \tan (x/2)}, \text{ if } a^2 < b^2.$$

81. $$\int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\sqrt{\frac{a - b}{a + b}} \tan \frac{x}{2} \right], \quad a^2 > b^2;$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b + a} + \sqrt{b - a} \tan (x/2)}{\sqrt{b + a} - \sqrt{b - a} \tan (x/2)}, \quad a^2 < b^2.$$

82. $$\int \frac{dx}{a \sin x + b \cos x} = \frac{1}{\sqrt{a^2 + b^2}} \log \tan \frac{x + \alpha}{2}, \quad \alpha = \sin^{-1} \frac{b}{\sqrt{a^2 + b^2}}.$$

NOTES. (*a*) $\int \frac{\cos x \, dx}{a + b \sin x}$, — use 79, (1), $= \frac{1}{b} \log (a + b \sin x)$.

(*b*) $\int \frac{A + B \sin x + C \cos x}{a + b \sin x} dx = C \int \frac{\cos x \, dx}{a + b \sin x} + \frac{B}{b} \int dx + \frac{Ab - Ba}{b} \int \frac{dx}{a + b \sin x}$, then use 82 *a*, 80.

(*c*) Many others similar to (*a*) and (*b*); *e.g.* $\int [\sin x / (a + b \cos x)] \, dx$, — use 79, (2).

(*d*) $\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$ and like forms, — use 79, (3); see 79, note *d*.

(*e*) As **last resort**, use 79, (5), for any rational trigonometric integral.

E_b. Integrand Transcendental: Trigonometric-Algebraic.

83. $$\int x^m \sin x \, dx = - x^m \cos x + m \int x^{m-1} \cos x \, dx.$$

84. $$\int x^m \cos x \, dx = x^m \sin x - m \int x^{m-1} \sin x \, dx.$$

NOTES. (*a*) $\int x \sin x \, dx = - x \cos x + \int \cos x \, dx$, — use 58.

(*b*) $\int x^m \sin x \, dx$, — repeat 83 to reach 58.

(*c*) $\int$ (Any Polynomial) $\sin x \, dx$, — split up and use 83.

(*d*) For $\cos x$ repeat (*a*), (*b*), (*c*).

85. $$\int \frac{\sin x \, dx}{x^m} = \frac{- \sin x}{(m - 1) x^{m-1}} + \frac{1}{m - 1} \int \frac{\cos x}{x^{m-1}} dx, \quad m \neq 1.$$

86. $$\int \frac{\cos x \, dx}{x^m} = - \frac{\cos x}{(m - 1) x^{m-1}} - \frac{1}{m - 1} \int \frac{\sin x}{x^{m-1}} dx, \quad m \neq 1.$$

87. $\int \frac{\sin x}{x}\,dx = \int \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right] dx$; see II, E, 13, p. 8.

88. $\int \frac{\cos x}{x}\,dx = \int \left[\frac{1}{x} - \frac{x}{2!} + \frac{x^3}{4!} - \cdots\right] dx$; see II, E, 14, p. 8.

NOTE. Other trigonometric-algebraic combinations, use 5; or 79 followed by 89–94.

E_c. Integrand Transcendental: Inverse Trigonometric.

89. $\int \sin^{-1} x\,dx = x \sin^{-1} x + \sqrt{1-x^2}$. [From 5.]

90. $\int \cos^{-1} x\,dx = x \cos^{-1} x - \sqrt{1-x^2}$.

91. $\int \tan^{-1} x\,dx = x \tan^{-1} x - \frac{1}{2}\log(1+x^2)$.

92. $\int x^n \sin^{-1} x\,dx = \frac{x^{n+1}\sin^{-1} x}{n+1} - \frac{1}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-x^2}}$, then 53 or 54, 32, 36.

93. $\int x^n \cos^{-1} x\,dx = \frac{x^{n+1}\cos^{-1} x}{n+1} + \frac{1}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-x^2}}$, then 53 or 54, 32, 36.

94. $\int x^n \tan^{-1} x\,dx = \frac{x^{n+1}\tan^{-1} x}{n+1} - \frac{1}{n+1}\int \frac{x^{n+1}\,dx}{1+x^2}$, then 19 (c).

NOTES. (a) Replace $\text{ctn}^{-1} x$ by $\frac{\pi}{2} - \tan^{-1} x$; or by $\tan^{-1}(1/x)$ and substitute $1/x = u$.
(b) Replace $\sec^{-1} x$ by $\cos^{-1}(1/x)$, $\csc^{-1} x$ by $\sin^{-1}(1/x)$ and substitute $1/x = u$.
(c) $\int$ (Any Polynomial) $\sin^{-1} x\,dx$, split up and use 92. (Similarly for $\cos^{-1} x$, etc.)
(d) $\int f(x) \sin^{-1} x\,dx$, – use (5) with $u = \sin^{-1} x$. (Similarly for $\cos^{-1} x$ and $\tan^{-1} x$.)
(e) Other Inverse Trigonometric Integrands, use 79 or 5.

E_d. Integrand Transcendental: Exponential and Logarithmic

95. $\int a^x dx = \frac{a^x}{\log_e a} = \frac{a^x}{\log_{10} a}\log_{10} e = \frac{a^x}{\log_{10} a}\,0.4343.$

96. $\int e^x\,dx = e^x.$

NOTES. (a) $\int \epsilon^{kx} dx = \epsilon^{kx} \div k$. (b) Notice $a^x = \epsilon^{(\log_e a)x} = \epsilon^{kx}$, $k = \log_e a$.

97. $\int x^n e^{kx}\,dx = \frac{1}{k} x^n e^{kx} - \frac{n}{k}\int x^{n-1} e^{kx}\,dx.$

NOTES. (a) $\int x e^{kx} dx = x e^{kx}/k - \epsilon^{kx}/k^2$. (b) $\int x^n \epsilon^{kx} dx$, — repeat 97 to reach 97 (a)
(c) $\int$ (Any Polynomial) $\epsilon^{kx} dx$, split up and use 97.

98. $\int \frac{e^{kx}}{x^m}\,dx = -\frac{e^{kx}}{(m-1)\,x^{m-1}} + \frac{k}{m-1}\int \frac{e^{kx}}{x^{m-1}}\,dx$ (repeat to reach 99).

99. $\int \frac{e^{kx}}{x}\,dx = \int \left[\frac{1}{u} + 1 + \frac{u}{2\,!} + \frac{u^2}{3\,!} + \cdots\right] du,\ u = kx$; see *Tables*, V, H

100. $\int e^{kx} \sin nx\,dx = e^{kx}\frac{k \sin nx - n \cos nx}{k^2 + n^2}.$

101. $\int e^{kx} \cos mx\,dx = e^{kx}\frac{k \cos mx + m \sin mx}{k^2 + m^2}.$

102. $\int \log x\,dx = x \log x - x.$

103. $\int (\log x)^n \frac{dx}{x} = \frac{(\log x)^{n+1}}{n+1},\ n \neq -1.$

104. $\int \frac{dx}{\log x} = \int \frac{e^u\,du}{u},\ u = \log x$; see 99 and *Tables*, V, H.

105. $\int x^n \log x\,dx = x^{n+1}\left[\frac{\log x}{n+1} - \frac{1}{(n+1)^2}\right].$

106. $\int e^{kx} \log x\,dx = \frac{1}{k}\,e^{kx} \log x - \frac{1}{k}\int \frac{e^{kx}}{x}\,dx$, see 99.

F. Some Important Definite Integrals.

107. $\int_1^\infty \frac{dx}{x^m} = \frac{1}{m-1}$, **if** $m > 1$ (otherwise non-existent).

108. $\int_0^\infty \frac{dx}{a^2 + b^2x^2} = \frac{\pi}{2\,ab}.$

109. $\int_0^\infty x^n e^{-x}\,dx = \Gamma(n+1) = n\,!$ if n is integral. See V, **F**, p. *54*.

NOTES. (*a*) In general, $\Gamma(n+1) = n \cdot \Gamma(n)$, as for $n\,!$, if $n > 0$.
(*b*) $\Gamma(2) = \Gamma(1) = 1,\ \Gamma(1/2) = \sqrt{\pi}.\quad \Gamma(n+1) = \Pi(n).$

110. $\int_0^1 x^m(1-x)^n\,dx = \frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)}.$

111. $\int_0^\pi \sin nx \cdot \sin mx\,dx = \int_0^\pi \cos nx \cos mx\,dx = 0$, **if** $m \neq n$, if m and n are integral.

112. $\int_0^\pi \sin^2 nx\,dx = \int_0^\pi \cos^2 nx\,dx = \pi/2$; n integral, see 56, 59.

113. $\int_0^\infty e^{-kx}\,dx = 1/k.$ 114. $\int_0^\infty [(\sin nx)/x]\,dx = \pi/2.$

115. $\int_0^\infty e^{-kx}\sin nx\,dx = n/(k^2+n^2),$ if $k>0.$

116. $\int_0^\infty e^{-kx}\cos mx\,dx = k/(k^2+m^2),$ if $k>0.$

117. $\int_0^\infty e^{-kx}x^n\,dx = \frac{\Gamma(n+1)}{k^{n+1}} = \frac{n!}{k^{n+1}},$ if n is integral. See 109.

118. $\int_0^\infty e^{-k^2x^2}\,dx = \sqrt{\pi}/(2k).$

119. $\int_0^\infty e^{-k^2x^2}\cos mx\,dx = \frac{\sqrt{\pi}\,e^{-m^2/4k^2}}{2k},$ if $k>0.$

120. $\int_0^\infty \frac{2\,dx}{e^{kx}+e^{-kx}} = \int_0^\infty \frac{dx}{\cosh kx} = \frac{\pi}{2k}.$ 121. $\int_0^1 (\log x)^n\,dx = (-1)^n\,n!$

122. $\int_0^{\pi/2}\log\sin x\,dx = \int_0^{\pi/2}\log\cos x\,dx = -\frac{\pi}{2}\log 2.$

123. $\int_0^{\pi/2}\sin^{2n+1}x\,dx = \int_0^{\pi/2}\cos^{2n+1}x\,dx = \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots(2n+1)}$ (n, positive integer.)

124. $\int_0^{\pi/2}\sin^{2n}x\,dx = \int_0^{\pi/2}\cos^{2n}x\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\,\frac{\pi}{2}$ (n, positive integer.)

G. Approximation Formulas.

125. $\int_a^b f(x)\,dx = f(c)(b-a), \quad a<c<b.$ [**Law of the Mean**.]

126. $\int_a^b f(x)dx = \frac{f(b)+f(a)}{2}(b-a).$ [**Trapezoid Rule** — precise for a straight line.]

127. $\int_a^b f(x)\,dx.$ [*Extended Trapezoid Rule.*]

$= [f(a)/2 + f(a+\Delta x) + f(a+2\Delta x) + \cdots + f[a+(n-1)\Delta x] + f(b)/2]\Delta x.$

128. $\int_a^b f(x)\,dx = \frac{f(a)+4f[(a+b)/2]+f(b)}{6}(b-a).$

[**Prismoid Rule**; or *second Simpson-Lagrange* approximation; precise if $f(x)$ is any quadratic or cubic; see § 71, p. 126.]

129. $\int_a^b f(x)dx = \frac{\Delta x}{3}[f(a) + 4f(a+\Delta x) + 2f(a+2\Delta x)$
$+ 4f(a+3\Delta x) + 2f(a+4\Delta x) + \cdots + f(b)].$

[**Simpson's Rule**; or extended prismoid rule. See § 125, p. 240.]

130. $\int_a^b f(x)\,dx$

$$= \frac{f(a) + 3f[a + \Delta x] + 3f[a + 2\,\Delta x] + f(b)}{8}(b - a)\,;\ \Delta x = (b - a)/3.$$

[A *third Simpson-Lagrange Approximation.* Extend as in 129.]

131. $\int_a^b f(x)\,dx$

$$= \frac{7f(a) + 32f[a + \Delta x] + 12f[a + 2\,\Delta x] + 32f[a + 3\,\Delta x] + 7f(b)}{90}(b - a);\ \Delta x = (b - a)/4.$$

[A *fourth Simpson-Lagrange Approximation;* see *Lagrange interpolation formula,* II, I, 17, p. *15.*]

H. Standard Applications of Integration.

132. *Areas of Plane Figures:* $\int dA$.

(*a*) Strips ΔA parallel to y-axis: $dA = y\,dx$.

(*b*) Strips ΔA parallel to x-axis: $dA = x\,dy$.

(*c*) Rectangles $\Delta A = \Delta x\,\Delta y$: $dA = dx\,dy$, $A = \iint dx\,dy$.

(*d*) Parameter form of equation: $A = (1/2)\int (x\,dy - y\,dx)$.

(*e*) Polar sectors bounded by radii: $dA = (\rho^2/2)\,d\theta$.

(*f*) Polar rectangles $\Delta A = \rho\,\Delta\rho\,\Delta\theta$: $dA = \rho\,d\rho\,d\theta$; $A = \iint \rho\,d\rho\,d\theta$.

133. *Lengths of Plane Curves:* $\int ds$.

(*a*) Equation in form $y = f(x)$: $ds = \sqrt{1 + [f'(x)]^2}\,dx$.

(*b*) Equation in form $x = \phi(y)$: $ds = \sqrt{1 + [\phi'(y)]^2}\,dy$.

(*c*) Parameter equations: $ds = \sqrt{dx^2 + dy^2}$.

(*d*) Polar equation: $ds = \sqrt{d\rho^2 + \rho^2\,d\theta^2}$.

134. *Volumes of Solids:* $\int dV$.

(*a*) *Frustum* (area of cross section A): $dV = A\,dh$; $V = \int A\,dh$ where h is the variable height perpendicular to the cross section A.

(*b*) Solid of revolution about x-axis: $dV = \pi y^2\,dx$.

(*c*) Solid of revolution about y-axis: $dV = \pi x^2\,dy$.

(*d*) Rectangular coördinate divisions: $dV = dx\,dy\,dz$;

$$V = \iiint dz\,dy\,dx = \iint z\,dy\,dx = \int\{\int z\,dy\}\,dx = \int A\,dx.$$

(*e*) Polar coördinate divisions. $dV = \rho^2 \sin\theta\,d\rho\,d\phi\,d\theta$.

135. *Area of a Surface:* $\int\int \sec\psi \, dx\, dy,$

where ψ is the angle between the element ds of the surface and its projection $dx\,dy$.

(*a*) Surface of Revolution about x-axis: $A = \int 2\pi y\, ds$.

(*b*) Surface of Revolution about y-axis: $A = \int 2\pi x\, ds$.

136. *Length of twisted arcs:* $\int ds.$

(*a*) Rectangular Coördinates: $ds = \sqrt{dx^2 + dy^2 + dz^2}$.

(*b*) Explicit Equations $y = f(x)$, $z = \phi(x)$: $ds = \sqrt{1 + [f'(x)]^2 + [\phi'(x)]^2}$.

(*c*) $x = f(t)$, $y = \phi(t)$, $z = \psi(t)$: $ds = \sqrt{[f'(t)]^2 + [\phi'(t)]^2 + [\psi'(t)]^2}$.

(*d*) Polar Coördinates: $ds = \sqrt{d\rho^2 + \rho^2 d\phi^2 + \rho^2 \cos^2\phi\, d\theta^2}$.

137. *Mass of a body:* $M = \int dM = \int \rho\, dV,$

where ρ is the density (mass per unit volume).

(*a*) If ρ is constant: $M = \rho \int dV$; see 134.

(*b*) On any curve: $dV = ds$, if ρ = mass per unit length.

(*c*) On any surface (or plane): $dV = dA$, if ρ = mass per unit area.

138. *Average value of a variable quantity q:* **A. V. of q.:**

(*a*) throughout a solid: $q = f(x, y, z)$; $A.\,V.$ of $q. = \int q\, dV \div \int dV$.

(*b*) on an area A: $A.\,V.$ of $q. = \int q\, dA \div \int dA$.

(*c*) on an arc s: $A.\,V.$ of $q. = \int q\, ds \div \int ds$.

139. *Center of Mass,* $(\bar{x}, \bar{y}, \bar{z})$: $\bar{x} = \int x\, dM \div \int dM,$

with similar formulas for $\bar{y}$ and $\bar{z}$. See dM, 137.

(*a*) for a volume: $dM = \rho\, dV$.

(*b*) for an area: $dM = \rho\, dA$.

(*c*) for an arc: $dM = \rho\, ds$.

139.* *Theorems* of *Pappus* or *Guldin:*

(*a*) Surface generated by an arc of a plane curve revolved about an axis in its plane = length of arc × length of path of center of mass of arc.

(*b*) Volume generated by revolving a closed plane contour about an axis in its plane = area of contour × length of path of its center of mass.

140. *Moment of Inertia:* $I = \int r^2\, dM.$ (See 137, 139.)

(*a*) For plane figures, $I_x + I_y = I_0$, where I_x, I_y, I_0 are taken about the x-axis, the y-axis, the origin, respectively.

(*b*) For space figures, $I_x + I_y + I_z = I_0$.

(*c*) $I_x = I_{\bar{x}} + (x - \bar{x})^2 M$, where $I_{\bar{x}}$ is taken about a line $\parallel$ to the x-axis.

141. *Radius of Gyration:* $k^2 = I \div M = \int r^2\,dM \div \int dM.$

[In 140 and 141, r may be the distance from some fixed point, or line, or plane.]

142. *Liquid pressure:* $p = \int \rho h\,dA,$

where p is the total pressure, dA is the elementary strip parallel to the surface; h is the depth below the surface; and ρ is the weight per unit volume of the liquid.

143. *Center of liquid pressure:* $\bar{h} = \int h^2 dA \div \int h dA.$

144. *Work of a variable force:* $W = \int f \cos \psi\,ds,$

where f is the numerical magnitude of the force, ds is the element of the arc of the path, and ψ is the angle between f and ds.

145. *Attraction exerted by a solid:* $F = k \int \frac{m\,dM}{r^2},$

where k is the attraction between two unit masses at unit distance, m is the attracted particle, dM is an element of the attracting body; r is the distance from m to dM.

Components F_x, F_y, F_z of F along Ox, Oy, Oz are:

$$F_x = km \int \frac{\cos\alpha\,dM}{r^2}, \quad F_y = km \int \frac{\cos\beta\,dM}{r^2}, \quad F_z = km \int \frac{\cos\gamma\,dM}{r^2},$$

where α, β, γ are the direction angles of a line joining m to dM.

146. *Work in an expanding gas:* $W = \int p\,dv.$

147. *Distance s, speed v, tangential acceleration j_T:*

$$j_T = \int v\,dt = \int \left\{ \int s\,dt \right\} dt.$$

[Similar forms for angular speed and acceleration.]

148. *Errors of observation:*

(*a*) Probability of an error between $x = a$ and $x = b$: $P = \int_{x=a}^{x=b} y\,dx$, where y is the probable number of errors of magnitude x.

(*b*) The usual formula $y = (h/\sqrt{\pi})\,e^{-h^2x^2}$ gives: $P = (h/\sqrt{\pi}) \int e^{-h^2x^2}\,dx$, where h is the so-called *measure of precision*.

(*c*) Probability of an error between $x = -a$ and $x = +a$: $P(a) = \int_{x=-a}^{x=a} y\,dx$.

(*d*) *Probable error* $= (0.477)/h =$ value of a for which $P(a) = 1/2$.

(*e*) *Mean error* $= \int_0^\infty xy\,dx \div \int_0^\infty y\,dx = 1/(h\sqrt{\pi})$

V. NUMERICAL TABLES

A. TRIGONOMETRIC FUNCTIONS

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.0000	0°	.0000	$-\infty$	.0000	$-\infty$	∞	∞	1.0000	0000	90°	1.5708
.0175	1°	.0175	2419	.0175	2419	57.290	7581	.9998	9999	89°	1.5533
.0349	2°	.0349	5428	.0349	5431	28.636	4569	.9994	9997	88°	1.5359
.0524	3°	.0523	7188	.0524	7194	19.081	2806	.9986	9994	87°	1.5184
.0698	4°	.0698	8436	.0699	8446	14.301	1554	.9976	9989	86°	1.5010
.0873	5°	.0872	9403	.0875	9420	11.430	0580	.9962	9983	85°	1.4835
.1047	6°	.1045	0192	.1051	0216	9.5144	9784	.9945	9976	84°	1.4661
.1222	7°	.1219	0859	.1228	0891	8.1443	9109	.9925	9968	83°	1.4486
.1396	8°	.1392	1436	.1405	1478	7.1154	8522	.9903	9958	82°	1.4312
.1571	9°	.1564	1943	.1584	1997	6.3138	8003	.9877	9946	81°	1.4137
.1745	10°	.1736	2397	.1763	2463	5.6713	7537	.9848	9934	80°	1.3963
.1920	11°	.1908	2806	.1944	2887	5.1446	7113	.9816	9919	79°	1.3788
.2094	12°	.2079	3179	.2126	3275	4.7046	6725	.9781	9904	78°	1.3614
.2269	13°	.2250	3521	.2309	3634	4.3315	6366	.9744	9887	77°	1.3439
.2443	14°	.2419	3837	.2493	3968	4.0108	6032	.9703	9869	76°	1.3265
.2618	15°	.2588	4130	.2679	4281	3.7321	5719	.9659	9849	75°	1.3090
.2793	16°	.2756	4403	.2867	4575	3.4874	5425	.9613	9828	74°	1.2915
.2967	17°	.2924	4659	.3057	4853	3.2709	5147	.9563	9806	73°	1.2741
.3142	18°	.3090	4900	.3249	5118	3.0777	4882	.9511	9782	72°	1.2566
.3316	19°	.3256	5126	.3443	5370	2.9042	4630	.9455	9757	71°	1.2392
.3491	20°	.3420	5341	.3640	5611	2.7475	4389	.9397	9730	70°	1.2217
.3665	21°	.3584	5543	.3839	5842	2.6051	4158	.9336	9702	69°	1.2043
.3840	22°	.3746	5736	.4040	6064	2.4751	3936	.9272	9672	68°	1.1868
.4014	23°	.3907	5919	.4245	6279	2.3559	3721	.9205	9640	67°	1.1694
.4189	24°	.4067	6093	.4452	6486	2.2460	3514	.9135	9607	66°	1.1519
.4363	25°	.4226	6259	.4663	6687	2.1445	3313	.9063	9573	65°	1.1345
.4538	26°	.4384	6418	.4877	6882	2.0503	3118	.8988	9537	64°	1.1170
.4712	27°	.4540	6570	.5095	7072	1.9626	2928	.8910	9499	63°	1.0996
.4887	28°	.4695	6716	.5317	7257	1.8807	2743	.8829	9459	62°	1.0821
.5061	29°	.4848	6856	.5543	7438	1.8040	2562	.8746	9418	61°	1.0647
.5236	30°	.5000	6990	.5774	7614	1.7321	2386	.8660	9375	60°	1.0472
.5411	31°	.5150	7118	.6009	7788	1.6643	2212	.8572	9331	59°	1.0297
.5585	32°	.5299	7242	.6249	7958	1.6003	2042	.8480	9284	58°	1.0123
.5760	33°	.5446	7361	.6494	8125	1.5399	1875	.8387	9236	57°	.9948
.5934	34°	.5592	7476	.6745	8290	1.4826	1710	.8290	9186	56°	.9774
.6109	35°	.5736	7586	.7002	8452	1.4281	1548	.8192	9134	55°	.9599
.6283	36°	.5878	7692	.7265	8613	1.3764	1387	.8090	9080	54°	.9425
.6458	37°	.6018	7795	.7536	8771	1.3270	1229	.7986	9023	53°	.9250
.6632	38°	.6157	7893	.7813	8928	1.2799	1072	.7880	8965	52°	.9076
.6807	39°	.6293	7989	.8098	9084	1.2349	0916	.7771	8905	51°	.8901
.6981	40°	.6428	8081	.8391	9238	1.1918	0762	.7660	8843	50°	.8727
.7156	41°	.6561	8169	.8693	9392	1.1504	0608	.7547	8778	49°	.8552
.7330	42°	.6691	8255	.9004	9544	1.1106	0456	.7431	8711	48°	.8378
.7505	43°	.6820	8338	.9325	9697	1.0724	0303	.7314	8641	47°	.8203
.7679	44°	.6947	8418	.9657	9848	1.0355	0152	.7193	8569	46°	.8029
.7854	45°	.7071	8495	1.0000	0000	1.0000	0000	.7071	8495	45°	.7854
		Value	$\log_{10}$	Value	$\log_{10}$	Value	$\log_{10}$	Value	$\log_{10}$	Degrees	Radians
		Cosine		Cotangent		Tangent		Sine			

B. COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9	D
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8

N	0	1	2	3	4	5	6	7	8	9	D
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

C. EXPONENTIAL AND HYPERBOLIC FUNCTIONS

		e^x		e^{-x}		sinh x		cosh x	
x	$\log_e x$	Value	$\log_{10}$	Value	$\log_{10}$	Value	$\log_{10}$	Value	$\log_{10}$
0.0	$-\infty$	1.000	0.000	1.000	0.000	0.000	$-\infty$	1.000	0
0.1	−2.303	1.105	0.043	0.905	9.957	0.100	9.001	1.005	0.002
0.2	−1.610	1.221	0.087	0.819	9.913	0.201	9.304	1.020	0.009
0.3	−1.204	1.350	0.130	0.741	9.870	0.305	9.484	1.045	0.019
0.4	−0.916	1.492	0.174	0.670	9.826	0.411	9.614	1.081	0.034
0.5	−0.693	1.649	0.217	0.607	9.783	0.521	9.717	1.128	0.052
0.6	−0.511	1.822	0.261	0.549	9.739	0.637	9.804	1.185	0.074
0.7	−0.357	2.014	0 304	0.497	9.696	0.759	9.880	1.255	0.099
0.8	−0.223	2.226	0.347	0.449	9.653	0.888	9.948	1.337	0.126
0.9	−0.105	2.460	0.391	0.407	9.609	1.027	0.011	1.433	0.156
1.0	0.000	2.718	0.434	0.368	9.566	1.175	0.070	1.543	0.188
1.1	0.095	3.004	0.478	0.333	9.522	1.336	0.126	1.669	0.222
1.2	0.182	3.320	0.521	0.301	9.479	1.509	0.179	1.811	0.258
1.3	0.262	3.669	0.565	0.273	9.435	1.698	0.230	1.971	0.295
1.4	0.336	4.055	0.608	0.247	9.392	1.904	0.280	2.151	0.333
1.5	0.405	4.482	0.651	0.223	9.349	2.129	0.328	2.352	0.372
1.6	0.470	4.953	0.695	0.202	9.305	2.376	0.376	2.577	0.411
1.7	0.531	5.474	0.738	0.183	9.262	2.646	0.423	2.828	0.452
1.8	0.588	6.050	0.782	0.165	9.218	2.942	0.469	3.107	0.492
1.9	0.642	6.686	0.825	0.150	9.175	3.268	0.514	3.418	0.534
2.0	0.693	7.389	0.869	0.135	9.131	3.627	0.560	3.762	0.575
2.1	0.742	8.166	0.912	0.122	9.088	4.022	0.604	4.144	0.617
2.2	0.788	9.025	0.955	0.111	9.045	4.457	0.649	4.568	0.660
2.3	0.833	9.974	0.999	0.100	9.001	4.937	0.690	5.037	0.702
2.4	0.875	11.02	1.023	0.091	8.958	5.466	0.738	5.557	0.745
2.5	0.916	12.18	1.086	0.082	8.914	6.050	0.782	6.132	0.788
2.6	0.956	13.46	.1.129	0.074	8.871	6.695	0.826	6.769	0.831
2.7	0.993	14.88	1.173	0.067	8.827	7.406	0.870	7.473	0.874
2.8	1.030	16.44	1.216	0.061	8.784	8.192	0.913	8.253	0.917
2.9	1.065	18.17	1.259	0.055	8.741	9.060	0.957	9.115	0.960
3.0	1.099	20.09	1.303	0.050	8.697	10.018	1.001	10.068	1.003
3.5	1.253	33.12	1.520	0.030	8.480	16.543	1.219	16.573	1.219
4.0	1.386	54.60	1.737	0.018	8.263	27.290	1.436	27.308	1.436
4.5	1.504	90.02	1.954	0.011	8.046	45.003	1.653	45.014	1.653
5.0	1.609	148.4	2.171	0.007	7.829	74.203	1.870	74.210	1.870
6.0	1.792	403.4	2.606	0.002	7.394	201.7	2.305	201.7	2.305
7.0	1.946	1096.6	3.040	0.001	6.960	548.3	2.739	548.3	2.739
8.0	2.079	2981.0	3.474	0.000	6.526	1490.5	3.173	1490.5	3.173
9.0	2.197	8103.1	3.909	0.000	6.091	4051.5	3.608	4051.5	3.608
10.0	2.303	22026.	4.343	0.000	5.657	11013.	4.041	11013.	4.041

$\log_e x = (\log_{10} x) \div M$; $M = .4342944819$. $\log_{10} e^{x+y} = \log_{10} e^x + \log_{10} e^y$.

Sinh x and cosh x approach $e^x/2$ as x increases (see Fig. E, p. 20). The formula $\log_{10} (e^x/2) = M \cdot x - \log_{10} 2$ represents $\log_{10}$ sinh x and $\log_{10}$ cosh x to three decimal places when $x > 3.5$; four places when $x > 5$; to five places when $x > 6$; to eight places when $x > 10$.

D. VALUES OF

$$F(k,\ \phi)=\int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\int_0^{u}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},\quad \left\{\begin{matrix} x=\sin\theta \\ u=\sin\phi \end{matrix}\right\}.$$

[Elliptic Integral of the First Kind.]

$k=$	$\phi=5^\circ$ $=\pi/36$	$\phi=10^\circ$ $=\pi/18$	$\phi=15^\circ$ $=\pi/12$	$\phi=30^\circ$ $=\pi/6$	$\phi=45^\circ$ $=\pi/4$	$\phi=60^\circ$ $=\pi/3$	$\phi=75^\circ$ $=5\pi/12$	K $\phi=90^\circ$ $=\pi/2$
0.0	0.087	0.175	0.262	0.524	0.785	1.047	1.309	1.571
0.1	0.087	0.175	0.262	0.524	0.786	1.049	1.312	1.575
0.2	0.087	0.175	0 262	0.525	0.789	1.054	1.321	1.588
0.3	0.087	0.175	0.262	0.526	0.792	1.062	1.336	1.610
0.4	0.087	0.175	0.262	0.527	0.798	1.074	1.358	1.643
0.5	0.087	0.175	0.263	0.529	0.804	1.090	1.385	1.686
0.6	0.087	0.175	0.263	0.532	0.814	1.112	1.426	1.752
0.7	0.087	0.175	0.263	0.536	0.826	1.142	1.488	1.854
0.8	0.087	0.175	0.264	0 539	0.839	1.178	1.566	1.993
0.9	0.087	0.175	0.264	0.544	0.858	1.233	1.703	2.275
1.0	0.087	0.175	0.265	0.549	0.881	1.317	2.028	∞

E. VALUES OF

$$E(k,\ \phi)=\int_0^{\phi}\sqrt{1-k^2\sin^2\theta}\,d\theta=\int_0^{u}\frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx,\quad \left\{\begin{matrix} x=\sin\theta \\ u=\sin\phi \end{matrix}\right\}.$$

[Elliptic Integral of the Second Kind.]

$k=$	$\phi=5^\circ$ $=\pi/36$	$\phi=10^\circ$ $=\pi/18$	$\phi=15^\circ$ $=\pi/12$	$\phi=30^\circ$ $=\pi/6$	$\phi=45^\circ$ $=\pi/4$	$\phi=60^\circ$ $=\pi/3$	$\phi=75^\circ$ $=5\pi/12$	E $\phi=90^\circ$ $=\pi/2$
0.0	0 087	0.175	0.262	0.524	0.785	1.047	1.309	1.571
0.1	0.087	0.175	0.262	0.523	0.785	1.046	1.306	1.566
0.2	0.087	0.174	0.262	0.523	0.782	1.041	1.297	1.554
0.3	0.087	0.174	0.262	0.521	0.779	1.033	1.283	1.533
0.4	0.087	0.174	0.261	0.520	0.773	1.026	1.264	1.504
0.5	0.087	0.174	0.261	0.518	0.767	1.008	1.240	1.467
0.6	0.087	0.174	0.261	0.515	0.759	0.989	1.207	1.417
0.7	0.087	0.174	0.260	0.512	0.748	0.965	1.163	1.351
0.8	0.087	0.174	0.260	0.509	0.737	0.940	1.117	1.278
0.9	0.087	0.174	0.259	0.505	0.723	0.907	1.053	1.173
1.0	0.087	0.174	0.259	0.500	0.707	0.866	0.966	1.000

F. VALUES OF $\Pi(p) = \Gamma(p+1) = \int_0^\infty e^{-x} x^p dx$

p A PROPER FRACTION

[$\Pi(n) = \Gamma(n+1) = n!$, if n is a positive integer.]

	$p=0.0$	$p=0.1$	$p=0.2$	$p=0.3$	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$	$p=$
$\Gamma(p+1)=$	1.000	0.951	0.918	0.897	0.887	$0.886=\sqrt{\pi}/2$	0.894	0.909	0.931	0.9

$\Gamma(k+1) = k\,\Gamma(k)$, if $k > 0$; hence $\Gamma(k+1)$ can be calculated at intervals of 0.1
Minimum value of $\Gamma(p+1)$ is .88560 at $p = .46163$.

G. VALUES OF THE PROBABILITY INTEGRAL: $\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} d$

x	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	.0000	.1125	.2227	.3286	.4284	.5205	.6039	.6778	.7421	.796
1.	.8427	.8802	.9103	.9340	.9523	.9661	.9763	.9838	.9891	.992
2.	.9953	.9970	.9981	.9989	.9993	.9996	.9998	.9999	.9999	1.000

H. VALUES OF THE INTEGRAL $\int_{-\infty}^{x} \frac{e^x dx}{x}$

[Note break at $x = 0$.]

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=$
$x=-n$ *	− .2194	− .0489	− .0130	− .0038	− .0012	− .0004	− .0001	− .0000	− .0
$x=-n/10$	− 1.823	− 1.223	− .9057	− .7024	− .5598	− .4544	− .3738	− .3106	− .2
$x=+n/10$	− 1.623	− .8218	− .3027	+ .1048	.4542	.7699	1.065	1.347	1.
$x=+n$	1.895	4.954	9.934	19.63	40.18	85.99	191.5	440.4	10

* Note. $\int_{-\infty}^{0} \frac{e^x dx}{x} = -\infty$. Values on each side of $x = 0$ can be used safely.

$\int_0^x \frac{dx}{\log x}$ and $\int \frac{e^{ax}}{x^n} dx$ reduce to the integral here tabulated; see IV, 99, 104, p.

I_1. RECIPROCALS OF NUMBERS FROM 1 TO 9.9

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	1.000	0.909	0.833	0.769	0.714	0.667	0.625	0.588	0.556	0.526
2	0.500	0.476	0.455	0.435	0.417	0.400	0.385	0.370	0.357	0.345
3	0.333	0.323	0.313	0.303	0.294	0.286	0.278	0.270	0.263	0.256
4	0.250	0.244	0.238	0.233	0.227	0.222	0.217	0.213	0.208	0.204
5	**0.200**	**0.196**	**0.192**	**0.189**	**0.185**	**0.182**	**0.179**	**0.175**	**0.172**	**0.169**
6	0.167	0.164	0.161	0.159	0.156	0.154	0.152	0.149	0.147	0.145
7	0.143	0.141	0.139	0.137	0.135	0.133	0.132	0.130	0.128	0.127
8	0.125	0.123	0.122	0.120	0.119	0.118	0.116	0.115	0.114	0.112
9	0.111	0.110	0.109	0.108	0.106	0.105	0.104	0.103	0.102	0.101

I_2. SQUARES OF NUMBERS FROM 10 TO 99

	0	1	2	3	4	5	6	7	8	9
1	100	121	144	169	196	225	256	289	324	361
2	400	441	484	529	576	625	676	729	784	841
3	900	961	1024	1089	1156	1225	1296	1369	1444	1521
4	1600	1681	1764	1849	1936	2025	2116	2209	2304	2401
5	**2500**	**2601**	**2704**	**2809**	**2916**	**3025**	**3136**	**3249**	**3364**	**3481**
6	3600	3721	3844	3969	4096	4225	4356	4489	4624	4761
7	4900	5041	5184	5329	5476	5625	5776	5929	6084	6241
8	6400	6561	6724	6889	7056	7225	7396	7569	7744	7921
9	8100	8281	8464	8649	8836	9025	9216	9409	9604	9801

I_3. CUBES OF NUMBERS FROM 1 TO 9.9

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	1.00	1.33	1.73	2.20	2.74	3.37	4.10	4.91	5.83	6.86
2	8.00	9.26	10.65	12.17	13.82	15.62	17.58	19.68	21.95	24.39
3	27.00	29.79	32.77	35.94	39.30	42.87	46.66	50.65	54.87	56.32
4	64.0	68.9	74.1	79.5	85.2	91.1	97.3	103.8	110.6	117.6
5	**125.0**	**132.7**	**140.6**	**148.9**	**157.5**	**166.4**	**175.6**	**185.2**	**195.1**	**205.4**
6	216.0	227.0	238.3	250.0	262.1	274.6	287.5	300.8	314.4	328.5
7	343.0	357.9	373.2	389.0	405.2	421.9	439.0	456.5	474.6	493.0
8	512.0	531.4	551.4	571.8	592.7	614.1	636.1	658.5	681.5	705.0
9	729.0	753.6	778.7	804.4	830.6	857.4	884.7	912.7	941.2	970.3

J_1. SQUARE ROOTS OF NUMBERS FROM 1 TO 9.9

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	0.000	0.316	0.447	0.548	0.632	0.707	0.775	0.837	0.894	0.949
1	1.000	1.049	1.095	1.140	1.183	1.225	1.265	1.304	1.342	1.378
2	1.414	1.449	1.483	1.517	1.549	1.581	1.612	1.643	1.673	1.703
3	1.732	1.761	1.789	1.817	1.844	1.871	1.897	1.924	1.949	1.975
4	2.000	2.025	2.049	2.074	2.098	2.121	2.145	2.168	2.191	2.214
5	**2.236**	**2.258**	**2.280**	**2.302**	**2.324**	**2.345**	**2.366**	**2.387**	**2.408**	**2.429**
6	2.449	2.470	2.490	2.510	2.530	2.550	2.569	2.588	2.608	2.627
7	2.646	2.665	2.683	2.702	2.720	2.739	2.757	2.775	2.793	2.811
8	2.828	2.846	2.864	2.881	2.898	2.915	2.933	2.950	2.966	2.983
9	3.000	3.017	3.033	3.050	3.066	3.082	3.098	3.114	3.130	3.146

J_2. SQUARE ROOTS OF NUMBERS FROM 10 TO 99

	0	1	2	3	4	5	6	7	8	9
1	3.162	3.317	3.464	3.606	3.742	3.873	4.000	4.123	4.243	4.359
2	4.472	4.583	4.690	4.796	4.899	5.000	5.099	5.196	5.292	5.385
3	5.477	5.568	5.657	5.745	5.831	5.916	6.000	6.083	6.164	6.245
4	6.325	6.403	6.481	6.557	6.633	6.708	6.782	6.856	6.928	7.000
5	**7.071**	**7.141**	**7.211**	**7.280**	**7.348**	**7.416**	**7.483**	**7.550**	**7.616**	**7.681**
6	7.746	7.810	7.874	7.937	8.000	8.062	8.124	8.185	8.246	8.307
7	8.367	8.426	8.485	8.544	8.602	8.660	8.718	8.775	8.832	8.888
8	8.944	9.000	9.055	9.110	9.165	9.220	9.274	9.327	9.381	9.434
9	9.487	9.539	9.592	9.644	9.695	9.747	9.798	9.849	9.899	9.950

K. RADIANS TO DEGREES

	RADIANS	TENTHS	HUNDREDTHS	THOUSANDTHS	TEN-THOUSANDTHS
1	57°17′44″.8	5°43′46″.5	0°34′22″.6	0° 3′26″.3	0° 0′20″.6
2	114°35′29″.6	11°27′33″.0	1° 8′45″.3	0° 6′52″.5	0° 0′41″.3
3	171°53′14″.4	17°11′19″.4	1°43′07″.9	0°10′18″.8	0° 1′01″.9
4	229°10′59″.2	22°55′05″.9	2°17′30″.6	0°13′45″.1	0° 1′22″.5
5	286°28′44″.0	28°38′52″.4	2°51′53″.2	0°17′11″.3	0° 1′43″.1
6	343°46′28″.8	34°22′38″.9	3°26′15″.9	0°20′37″.6	0° 2′03″.8
7	401° 4′13″.6	40° 6′25″.4	4° 0′38″.5	0°24′03″.9	0° 2′24″.4
8	458°21′58″.4	45°50′11″.8	4°35′01″.2	0°27′30″.1	0° 2′45″.0
9	515°39′43″.3	51°33′58″.3	5° 9′23″.8	0°30′56″.4	0° 3′05″.6

L. IMPORTANT CONSTANTS AND THEIR COMMON LOGARITHMS

N = Number	Value of N	$\text{Log}_{10} N$
π	3.14159265	0.49714987
$1 \div \pi$	0.31830989	9.50285013
π^2	9.86960440	0.99429975
$\sqrt{\pi}$	1.77245385	0.24857494
e = Napierian Base	2.71828183	0.43429448
$M = \log_{10} e$	0.43429448	9.63778431
$1 \div M = \log_e 10$	2.30258509	0.36221569
$180 \div \pi$ = degrees in 1 radian	57.2957795	1.75812262
$\pi \div 180$ = radians in 1°	0.01745329	8.24187738
$\pi \div 10800$ = radians in 1′	0.0002908882	6.4637261
$\pi \div 648000$ = radians in 1″	0.000004848136811095	4.68557487
sin 1″	0.000004848136811076	4.68557487
tan 1″	0.000004848136811152	4.68557487
centimeters in 1 ft.	30.480	1.4840158
feet in 1 cm.	0.032808	8.5159842
inches in 1 m.	39.37	1.5951654
pounds in 1 kg.	2.20462	0.3433340
kilograms in 1 lb.	0.453593	9.6566660
g	32.16 ft./sec./sec.	1.5073
	= 981 cm./sec./sec.	2.9916690
weight of 1 cu. ft. of water	62.425 lb. (max. density)	1.7953+
weight of 1 cu. ft. of air	0.0807 lb. (at 32° F.)	8.907
cu. in. in 1 (U. S.) gallon	231	2 3636120
ft. lb. per sec. in 1 H. P.	550.	2.7403627
kg. m. per sec. in 1 H. P.	76.0404	1.8810445
watts in 1 H. P.	745.957	2.8727135

M. DEGREES TO RADIANS

1°	.01745	10°	.17453	100°	1.74533	6′	.00175	6″	.00003
2°	.03491	20°	.34907	110°	1.91986	7′	.00204	7″	.00003
3°	.05236	30°	.52360	120°	2.09440	8′	.00233	8″	.00004
4°	.06981	40°	.69813	130°	2.26893	9′	.00262	9″	.00004
5°	.08727	50°	.87266	140°	2.44346	10′	.00291	10″	.00005
6°	.10472	60°	1.04720	150°	2.61799	20′	.00582	20″	.00010
7°	.12217	70°	1.22173	160°	2.79253	30′	.00873	30″	.00015
8°	.13963	80°	1.39626	170°	2.96706	40′	.01164	40″	.00019
9°	.15708	90°	1.57080	180°	3.14159	50′	.01454	50″	.00024

N. SHORT CONVERSION TABLES AND OTHER DATA: MULTIPLES, POWERS, ETC., FOR VARIOUS NUMBERS

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$\pi \cdot n$	3.1416	6.2832	9.4248	12.566	15.708	18.850	21.991	25.133	28.274
$\pi \cdot n^2/4$	.78540	3.1416	7.0686	12.566	19.635	28.274	38.485	50.265	63.617
$\pi \cdot n^3/6$	.52360	4.1888	14.137	33.510	65.450	113.10	179.59	268.08	381.70
$\pi \div n$	3.1416	1.5708	1.0472	.78540	.62382	.52360	.44880	.39270	.34907
$n \div \pi$	.31831	.63662	.95493	1.2732	1.5915	1.9099	2.2282	2.5465	2.8648
$(\pi/180) \cdot n$	.01745	.03491	.05236	.06981	.08727	.10472	.12217	.13963	.15708
$(180/\pi) \cdot n$	57.296	114.59	171 89	229.18	286.48	343.77	401.07	458.37	515.66
$e \cdot n$	2.7183	5.4366	8.1548	10.873	13.591	16.310	19.028	21.746	24.465
$M \cdot n$	.43429	.86859	1.3028	1.7371	2.1714	2.6057	3.0400	3.4744	3.9087
$(1 \div M) \cdot n$	2.3026	4.6052	6.9078	9.2103	11.513	13.816	16.118	18.421	20.723
$1 \div n$	1.0000	.50000	.33333	.25000	.20000	.16667	.14286	.12500	.11111
n^2	1.	4.	9.	16.	25.	36.	49.	64.	81.
n^3	1.	8.	27.	84	125.	216.	343.	512.	729.
n^4	1.	16.	81.	256.	625.	1296.	2401.	4096.	6561.
n^5	1.	32.	243.	1024.	3125.	7776.	16807.	32768.	59049.
$2^5 \cdot 2^n$	64.	128.	256.	512.	1024.	2048.	4096.	8192.	16384.
3^n	3.	9.	27.	81.	243.	729.	2187.	6561.	19683.
$\sqrt{n}$	1.	1.4142	1.7321	2.	2.2361	2.4495	2.6458	2.8284	3.
$\sqrt[3]{n}$	1.	1.2599	1.4422	1.5874	1.7100	1.8171	1.9129	2.	2.0801
$n!$	1.	2.	6.	24.	120.	720.	5040.	40320.	362880.
$1 \div n!$	1.	0.5	.16667	.04167	.00833	.00139	.00020	.00002	.000003
B_n*	$1 \div 6$	1.	$1 \div 30$	5.	$1 \div 42$	61.	$1 \div 30$	1385.	$5 \div 66$
cm. in n in.	2.5400	5.0800	7.6200	10.160	12.700	15.240	17.780	20.320	22.860
in. in n cm.	.39370	.78740	1.1811	1.5748	1.9685	2.3622	2.7559	3.1496	3.5433†
m. in n ft.	.30480	.60960	.91440	1.2192	1.5240	1.8288	2.1336	2.4384	2.7432
ft. in n m.	3.2808	6.5617	9.8425	13.123	16.404	19.685	22.966	26.247	29.527
km. in n mi.	1.6093	3.2187	4.8280	6.4374	8.0467	9.6561	11.265	12.875	14.484
mi. in n km.	0.6214	1.2427	1.8641	2.4855	3.1069	3.7282	4.3496	4.9710	5.5923
kg. in n lb.	.45359	.90719	1.3608	1.8144	2.2680	2.7216	3.1751	3.6287	4.0823
lb. in n kg.	2.2046	4.4092	6.6139	8.8185	11.023	13.228	15.432	17.637	19.842
l. in n qt.	.94636	1.8927	2.8391	3.7854	4.7318	5.6782	6.6245	7.5709	8.5172
qt. in n l.	1.0567	2.1134	3.1700	4.2267	5.2834	6.3401	7.3968	8.4534	9.5101

* $B_n \equiv n$th Bernoulli number; see II, E, 15–18, p. 8.

† Exact legal values in U. S.

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